

ISOSPECTRAL COMMUTING VARIETY, THE HARISH-CHANDRA \mathcal{D} -MODULE, AND PRINCIPAL NILPOTENT PAIRS

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ABSTRACT. Let \mathfrak{g} be a complex reductive Lie algebra with Cartan algebra \mathfrak{t} . Hotta and Kashiwara defined a holonomic \mathcal{D} -module \mathcal{M} , on $\mathfrak{g} \times \mathfrak{t}$, called Harish-Chandra module. We relate $\mathrm{gr} \mathcal{M}$, an associated graded module with respect to a canonical *Hodge filtration* on \mathcal{M} , to the *isospectral commuting variety*, a subvariety of $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{t} \times \mathfrak{t}$ which is a ramified cover of the variety of pairs of commuting elements of \mathfrak{g} . Our main result establishes an isomorphism of $\mathrm{gr} \mathcal{M}$ with the structure sheaf of $\mathfrak{X}_{\mathrm{norm}}$, the normalization of the isospectral commuting variety. We deduce, using Saito's theory of Hodge \mathcal{D} -modules, that the scheme $\mathfrak{X}_{\mathrm{norm}}$ is Cohen-Macaulay and Gorenstein. This confirms a conjecture of M. Haiman.

Associated with any *principal nilpotent pair* in \mathfrak{g} , there is a finite subscheme of $\mathfrak{X}_{\mathrm{norm}}$. The corresponding coordinate ring is a bigraded finite dimensional Gorenstein algebra that affords the regular representation of the Weyl group. The socle of that algebra is a 1-dimensional space generated by a remarkable W -harmonic polynomial on $\mathfrak{t} \times \mathfrak{t}$. In the special case where $\mathfrak{g} = \mathfrak{gl}_n$ the above algebras are closely related to the $n!$ -theorem of Haiman and our W -harmonic polynomial reduces to the Garsia-Haiman polynomial. Furthermore, in the \mathfrak{gl}_n case, the sheaf $\mathrm{gr} \mathcal{M}$ gives rise to a vector bundle on the Hilbert scheme of n points in \mathbb{C}^2 that turns out to be isomorphic to the *Procesi bundle*. Our results were used by I. Gordon to obtain a new proof of positivity of the Kostka-Macdonald polynomials established earlier by Haiman.

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1. INTRODUCTION

1.1. Notation. We work over the ground field \mathbb{C} of complex numbers and we write $\otimes = \otimes_{\mathbb{C}}$.¹

By a scheme X we mean a scheme of finite type over \mathbb{C} . We write X_{red} for the corresponding reduced scheme and $\psi : X_{\mathrm{norm}} \rightarrow X_{\mathrm{red}}$ for the normalization map (if X_{red} is irreducible). Let \mathcal{O}_X denote the structure sheaf of X , resp. \mathcal{K}_X the canonical sheaf (if X is Cohen-Macaulay), and \mathcal{D}_X the sheaf of algebraic differential operators on X (if X is smooth). Write $\mathbb{C}[X] = \Gamma(X, \mathcal{O}_X)$, resp. $\mathcal{D}(X) = \Gamma(X, \mathcal{D}_X)$, for the algebra of global sections. Let T^*X denote (the total space of) the cotangent bundle on a smooth variety X .

¹A more complete Index of Notation is given at the end of the paper.

Given an algebraic group K and a K -action on E , we write E^K for the set of K -fixed points. In particular, for a K -variety X , one has the subalgebra $\mathbb{C}[X]^K \subset \mathbb{C}[X]$, resp. $\mathcal{D}(X)^K \subset \mathcal{D}(X)$, of K -invariants.

Throughout the paper, we fix a connected complex reductive group G with Lie algebra \mathfrak{g} . Let $T \subset G$ be a maximal torus, $\mathfrak{t} = \text{Lie } T$ the corresponding Cartan subalgebra of \mathfrak{g} , and $r = \dim \mathfrak{t}$ the rank of \mathfrak{g} . Write $N(T)$ for the normalizer of T in G so, $W = N(T)/T$ is the Weyl group. The group W acts on \mathfrak{t} via the reflection representation and it acts on $\wedge^r \mathfrak{t}$ by the sign character $w \mapsto \text{sign}(w)$. We write E^{sign} for the *sign*-isotypic component of a W -module E .

1.2. Definition of the Harish-Chandra module. We will use a special notation $\mathcal{D} := \mathcal{D}_{\mathfrak{g} \times \mathfrak{t}}$ for the sheaf of differential operators on $\mathfrak{g} \times \mathfrak{t}$. We have $\Gamma(\mathfrak{g} \times \mathfrak{t}, \mathcal{D}) = \mathcal{D}(\mathfrak{g} \times \mathfrak{t}) = \mathcal{D}(\mathfrak{g}) \otimes \mathcal{D}(\mathfrak{t})$ where $\mathcal{D}(\mathfrak{g})$, resp. $\mathcal{D}(\mathfrak{t})$, is the algebra of polynomial differential operators on the vector space \mathfrak{g} , resp. on \mathfrak{t} . The subalgebra of $\mathcal{D}(\mathfrak{g})$, resp. of $\mathcal{D}(\mathfrak{t})$, formed by the differential operators with constant coefficients may be identified with $\text{Sym } \mathfrak{g}$, resp. with $\text{Sym } \mathfrak{t}$, the corresponding symmetric algebra.

Let the group G act on \mathfrak{g} by the adjoint action. Harish-Chandra [HC] defined a ‘radial part’ map $\text{rad} : \mathcal{D}(\mathfrak{g})^G \rightarrow \mathcal{D}(\mathfrak{t})^W$. This is an algebra homomorphism such that its restriction to the subalgebra of G -invariant polynomials, resp. of G -invariant constant coefficient differential operators, reduces to the Chevalley isomorphism $\mathbb{C}[\mathfrak{g}]^G \xrightarrow{\sim} \mathbb{C}[\mathfrak{t}]^W$, resp. $(\text{Sym } \mathfrak{g})^G \xrightarrow{\sim} (\text{Sym } \mathfrak{t})^W$.

Given $a \in \mathfrak{g}$, one may view the map $\text{ad } a : \mathfrak{g} \rightarrow \mathfrak{g}$, $x \mapsto [a, x]$ as a (linear) vector field on \mathfrak{g} , that is, as a first order differential operator on \mathfrak{g} . The assignment $a \mapsto \text{ad } a$ gives a linear map $\text{ad} : \mathfrak{g} \rightarrow \mathcal{D}(\mathfrak{g})$ with image $\text{ad } \mathfrak{g}$. Thus, one can form a left ideal $\mathcal{D}(\text{ad } \mathfrak{g} \otimes 1) \subset \mathcal{D}$.

Definition 1.2.1. The Harish-Chandra module is a left \mathcal{D} -module defined as follows

$$\mathcal{M} := \mathcal{D} / (\mathcal{D}(\text{ad } \mathfrak{g} \otimes 1) + \mathcal{D}\{u \otimes 1 - 1 \otimes \text{rad}(u), u \in \mathcal{D}(\mathfrak{g})^G\}). \quad (1.2.2)$$

Remark 1.2.3. For a useful interpretation of this formula see also (2.4.1). \diamond

According to an important result of Hotta and Kashiwara [HK1], the Harish-Chandra module is a simple holonomic \mathcal{D} -module of ‘geometric origin’, cf. Lemma 2.4.3(ii) below. This implies that \mathcal{M} comes equipped with a natural structure of *Hodge module* in the sense of M. Saito [Sa]. In particular, there is a canonical Hodge filtration on \mathcal{M} , see §2.5. Taking an associated graded sheaf with respect to the Hodge filtration produces a coherent sheaf $\widetilde{\text{gr}}^{\text{Hodge}} \mathcal{M}$ on $T^*(\mathfrak{g} \times \mathfrak{t})$.

The support of the sheaf $\widetilde{\text{gr}}^{\text{Hodge}} \mathcal{M}$ turns out to be closely related to the commuting scheme of the Lie algebra \mathfrak{g} , see Theorem 1.3.3 below. The main idea of the paper is to exploit the powerful theory of Hodge modules to deduce new results concerning commuting schemes using information about the sheaf $\widetilde{\text{gr}}^{\text{Hodge}} \mathcal{M}$.

Remark 1.2.4. Definition 1.2.1 was motivated by, but is not identical to, the definition of Hotta and Kashiwara, see [HK1], formula (4.5.1). The equivalence of the two definitions follows from Remark 4.1.2, of §4.1 below.

1.3. Main results. Put $\mathfrak{G} = \mathfrak{g} \times \mathfrak{g}$ and let G act diagonally on \mathfrak{G} . The *commuting scheme* \mathfrak{C} is defined as the *scheme-theoretic* zero fiber of the commutator map $\kappa : \mathfrak{G} \rightarrow \mathfrak{g}$, $(x, y) \mapsto [x, y]$. Thus, \mathfrak{C} is a G -stable closed subscheme of \mathfrak{G} ; set-theoretically, one has $\mathfrak{C} = \{(x, y) \in \mathfrak{G} \mid [x, y] = 0\}$. The scheme \mathfrak{C} is known to be generically reduced and irreducible, cf. Proposition 2.1.1 below. It is a long standing open problem whether or not this scheme is reduced.

Let $\mathfrak{T} := \mathfrak{t} \times \mathfrak{t} \subset \mathfrak{G}$. It is clear that \mathfrak{T} is an $N(T)$ -stable closed subscheme of \mathfrak{C} and the resulting $N(T)$ -action on \mathfrak{T} factors through the diagonal action of the Weyl group $W = N(T)/T$. Therefore, restriction of polynomial functions gives algebra maps

$$\text{res} : \mathbb{C}[\mathfrak{G}]^G \rightarrow \mathbb{C}[\mathfrak{C}]^G \rightarrow \mathbb{C}[\mathfrak{T}]^W. \quad (1.3.1)$$

The *isospectral commuting variety* is defined to be the algebraic set:

$$\mathfrak{X} = \{(x_1, x_2, t_1, t_2) \in \mathfrak{C} \times \mathfrak{T} \mid P(x_1, x_2) = (\text{res } P)(t_1, t_2), \forall P \in \mathbb{C}[\mathfrak{C}]^G\}. \quad (1.3.2)$$

We view \mathfrak{X} as a *reduced* closed subscheme of $\mathfrak{G} \times \mathfrak{T}$, cf. also Definition 2.1.4.

To proceed further, we fix an invariant bilinear form $\langle -, - \rangle : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$. This gives an isomorphism $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$, $x \mapsto \langle x, - \rangle$, resp. $\mathfrak{t} \xrightarrow{\sim} \mathfrak{t}^*$, $t \mapsto -\langle t, - \rangle$ (the minus sign in the last formula is related to the minus sign that appears in the anti-involution $v \mapsto v^\top$ considered in §2.4). Thus, one gets an identification $T^*(\mathfrak{g} \times \mathfrak{t}) = \mathfrak{G} \times \mathfrak{T}$ so one may view $\widetilde{\text{gr}}^{\text{Hodge}} \mathcal{M}$ as a coherent sheaf on $\mathfrak{G} \times \mathfrak{T}$.

One of the main results of the paper, whose proof will be completed in §4.5, reads

Theorem 1.3.3. *There is a natural $\mathcal{O}_{\mathfrak{G} \times \mathfrak{T}}$ -module isomorphism $\psi_* \mathcal{O}_{\mathfrak{X}_{\text{norm}}} \xrightarrow{\sim} \widetilde{\text{gr}}^{\text{Hodge}} \mathcal{M}$.*

This theorem combined with some deep results of Saito [Sa], to be reviewed in §2.3, yields the following theorem that confirms a conjecture of M. Haiman, [Ha3, Conjecture 7.2.3].

Theorem 1.3.4. *$\mathfrak{X}_{\text{norm}}$ is a Cohen-Macaulay and Gorenstein variety with trivial canonical sheaf.*

Theorem 1.3.4 will be deduced from Theorem 1.3.3 in §2.6.

Corollary 1.3.5 (§6.4). *The scheme $\mathfrak{C}_{\text{norm}}$ is Cohen-Macaulay.*

Corollary 1.3.6 (§9.2). *The $\mathcal{D}_{\mathfrak{g}}$ -module $\mathcal{D}_{\mathfrak{g}}/\mathcal{D}_{\mathfrak{g}} \cdot \text{ad } \mathfrak{g}$ comes equipped with a canonical filtration F such that one has an isomorphism $\widetilde{\text{gr}}^F(\mathcal{D}_{\mathfrak{g}}/\mathcal{D}_{\mathfrak{g}} \cdot \text{ad } \mathfrak{g}) \cong \psi_* \mathcal{O}_{\mathfrak{C}_{\text{norm}}}$, of $\mathcal{O}_{\mathfrak{G}}$ -modules.*

The last corollary implies (see §9.2) the following result that has been proved earlier by Levasseur and Stafford [LS2, Theorem 1.2] in a totally different way.

Corollary 1.3.7. (i) $\mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g}) \cdot \text{ad } \mathfrak{g}$ is a Cohen-Macaulay (non-holonomic) left $\mathcal{D}(\mathfrak{g})$ -module.
(ii) *The natural right action of the algebra $\mathbb{C}[\mathfrak{g}]^G$ makes $\mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g}) \cdot \text{ad } \mathfrak{g}$ a flat right $\mathbb{C}[\mathfrak{g}]^G$ -module.*

1.4. Group actions. One has a natural $G \times W$ -action on $\mathfrak{g} \times \mathfrak{t}$, resp. on $\mathfrak{G} \times \mathfrak{T}$, where the group G acts on the first factor and the group W acts on the second factor. There is also a \mathbb{C}^\times -action on \mathfrak{g} , resp. on $\mathfrak{g} \times \mathfrak{t}$, by dilations. So, we obtain a $\mathbb{C}^\times \times \mathbb{C}^\times$ -action on $\mathfrak{G} = T^*\mathfrak{g}$, resp. on $\mathfrak{G} \times \mathfrak{T} = (\mathfrak{g} \times \mathfrak{t}) \times (\mathfrak{g} \times \mathfrak{t}) = T^*(\mathfrak{g} \times \mathfrak{t})$, such that the standard \mathbb{C}^\times -action by dilations along the fibers of the cotangent bundle corresponds, via the above identification, to the action of the subgroup $\{1\} \times \mathbb{C}^\times \subset \mathbb{C}^\times \times \mathbb{C}^\times$. Thus, we have made the space $\mathfrak{g} \times \mathfrak{t}$ a $G \times W \times \mathbb{C}^\times$ -variety, resp. the space $\mathfrak{G} \times \mathfrak{T}$ a $G \times W \times \mathbb{C}^\times \times \mathbb{C}^\times$ -variety.

The scheme \mathfrak{X} is clearly $G \times W \times \mathbb{C}^\times \times \mathbb{C}^\times$ -stable. The resulting $G \times W \times \mathbb{C}^\times \times \mathbb{C}^\times$ -action on \mathfrak{X} induces one on $\mathfrak{X}_{\text{norm}}$ since a reductive group action can always be lifted canonically to the normalization, [Kr], §4.4.

On the other hand, the group $G \times W \times \mathbb{C}^\times$ acts on $\mathfrak{g} \times \mathfrak{t}$ and the Harish-Chandra module \mathcal{M} has the natural structure of a $G \times W \times \mathbb{C}^\times$ -equivariant \mathcal{D} -module. The Hodge filtration on \mathcal{M} is canonical, therefore, this $G \times W \times \mathbb{C}^\times$ -action respects the filtration. Hence, the group $G \times W \times \mathbb{C}^\times$ acts naturally on $\widetilde{\text{gr}}^{\text{Hodge}} \mathcal{M}$. There is also an additional \mathbb{C}^\times -action on $\widetilde{\text{gr}}^{\text{Hodge}} \mathcal{M}$ that comes from the grading. Thus, combining all these actions together, one may view $\widetilde{\text{gr}}^{\text{Hodge}} \mathcal{M}$ as a $G \times W \times \mathbb{C}^\times \times \mathbb{C}^\times$ -equivariant coherent sheaf on $\mathfrak{G} \times \mathfrak{T}$.

The isomorphism of Theorem 1.3.3 respects the $G \times W \times \mathbb{C}^\times \times \mathbb{C}^\times$ -equivariant structures. The equivariant structure on the sheaf $\mathcal{O}_{\mathfrak{X}_{\text{norm}}}$ makes the coordinate ring $\mathbb{C}[\mathfrak{X}_{\text{norm}}]$ a bigraded locally finite $G \times W$ -module.

In §§3.4,4.5, we will construct a “DG resolution” of $\mathfrak{X}_{\text{norm}}$, a (derived) “double analogue” of the Grothendieck-Springer resolution, cf. Remark 4.5.4. We show that the DG algebra of global sections of our resolution is acyclic in nonzero degrees. Using this, a standard application of the Atiyah-Bott-Lefschetz fixed point theorem yields the following result.

Theorem 1.4.1 (see [BG, §2.4]). *The bigraded T -character of $\mathbb{C}[\mathfrak{X}_{\text{norm}}]$ is given by the formula*

$$\chi^{\mathbb{C}^\times \times \mathbb{C}^\times \times T}(\mathcal{O}_{\mathfrak{X}_{\text{norm}}}) = \frac{1}{(1-q)^{\mathbf{r}}(1-t)^{\mathbf{r}}} \cdot \sum_{w \in W} w \left(\prod_{\alpha \in R^+} \frac{1 - qt e^\alpha}{(1 - e^{-\alpha})(1 - q e^\alpha)(1 - t e^\alpha)} \right).$$

Here, R^+ denotes the set of positive roots of \mathfrak{g} and the product in the right hand side of the formula is understood as a formal power series of the form $\sum_{m,n \geq 0} a_{m,n} \cdot q^m t^n$ where the coefficients $a_{m,n}$ are viewed as elements of the representation ring of the torus T .

We refer to [BG] for the proof and some combinatorial applications of the above theorem.

1.5. A coherent sheaf on the commuting scheme. The first projection $\mathfrak{G} \times \mathfrak{T} \rightarrow \mathfrak{G}$ restricts to a map $p : \mathfrak{X} \rightarrow \mathfrak{C}$. Therefore, the composite $\mathfrak{X}_{\text{norm}} \rightarrow \mathfrak{X} \rightarrow \mathfrak{C}$ factors through a morphism $p_{\text{norm}} : \mathfrak{X}_{\text{norm}} \rightarrow \mathfrak{C}_{\text{norm}}$. It is immediate to see that p_{norm} is a *finite* $G \times \mathbb{C}^\times \times \mathbb{C}^\times$ -equivariant morphism and that the group W acts along the fibers of this morphism.

Let $\mathcal{R} := (p_{\text{norm}})_* \mathcal{O}_{\mathfrak{X}_{\text{norm}}}$. This is a $G \times \mathbb{C}^\times \times \mathbb{C}^\times$ -equivariant coherent sheaf of $\mathcal{O}_{\mathfrak{C}_{\text{norm}}}$ -algebras. The action of the group W along the fibers of the map p_{norm} gives a W -action on \mathcal{R} by $\mathcal{O}_{\mathfrak{C}_{\text{norm}}}$ -algebra automorphisms. Therefore, for any finite dimensional W -representation E , one has a coherent sheaf $\mathcal{R}^E := (E \otimes \mathcal{R})^W$, the E -isotypic component of \mathcal{R} . Equivalently, in terms of the contragredient representation E^* , we have $\mathcal{R}^E = \text{Hom}_W(E^*, \mathcal{R})$.

The sheaf \mathcal{R} enjoys the following properties, see §6.6.

Corollary 1.5.1. (i) *The sheaf \mathcal{R} is Cohen-Macaulay and we have*

$$\mathcal{O}_{\mathfrak{C}_{\text{norm}}} \cong \mathcal{R}^W, \quad \text{resp.} \quad \mathcal{K}_{\mathfrak{C}_{\text{norm}}} \cong \mathcal{R}^{\text{sign}}.$$

(ii) *There is a $G \times W \times \mathbb{C}^\times \times \mathbb{C}^\times$ -equivariant isomorphism $\mathcal{R} \cong \text{Hom}_{\mathcal{O}_{\mathfrak{C}_{\text{norm}}}}(\mathcal{R}, \mathcal{K}_{\mathfrak{C}_{\text{norm}}})$. Furthermore, for any finite dimensional W -module E , this gives an isomorphism*

$$\mathcal{R}^{E^* \otimes \text{sign}} \cong \text{Hom}_{\mathcal{O}_{\mathfrak{C}_{\text{norm}}}}(\mathcal{R}^E, \mathcal{K}_{\mathfrak{C}_{\text{norm}}}).$$

Given $x \in \mathfrak{g}$, let \mathfrak{g}_x denote the centralizer of x in \mathfrak{g} . Similarly, write $\mathfrak{g}_{x,y} = \mathfrak{g}_x \cap \mathfrak{g}_y$ for the centralizer of a pair $(x, y) \in \mathfrak{C}$ in \mathfrak{g} . We call an element $x \in \mathfrak{g}$, resp. a pair $(x, y) \in \mathfrak{C}$, *regular* if we have $\dim \mathfrak{g}_x = \mathbf{r}$, resp. $\dim \mathfrak{g}_{x,y} = \mathbf{r}$. Let \mathfrak{g}^r , resp. \mathfrak{C}^r , be the set of regular elements of \mathfrak{g} , resp. of \mathfrak{C} . One shows that the set \mathfrak{C}^r is a Zariski open and dense subset of \mathfrak{C} which is equal to the smooth locus of the scheme \mathfrak{C} , cf. Proposition 2.1.1 below.

There is a coherent sheaf \mathbf{g} on $\mathfrak{C}_{\text{norm}}$, the “universal stabilizer sheaf”, such that the geometric fiber of \mathbf{g} at each point is the Lie algebra of the isotropy group of that point under the G -action, cf. §5.4 for a more rigorous definition. Any pair $(x, y) \in \mathfrak{C}^r$ may be viewed as a point of $\mathfrak{C}_{\text{norm}}$. The sheaf $\mathbf{g}|_{\mathfrak{C}^r}$ is locally free; its fiber at any point $(x, y) \in \mathfrak{C}^r$ equals, by definition, the vector space $\mathfrak{g}_{x,y}$.

Part (i) of the following theorem says that the sheaf \mathcal{R} gives an algebraic vector bundle on \mathfrak{C}^r that has very interesting structures. Part (ii) of the theorem provides a description of the isotypic components $\mathcal{R}^{\wedge^s \mathfrak{t}} = (\wedge^s \mathfrak{t} \otimes \mathcal{R})^W$, corresponding to the wedge powers $\wedge^s \mathfrak{t}$, $s \geq 0$, of the reflection representation of W , in terms of the sheaf \mathbf{g} .

Theorem 1.5.2 (§§6.4,6.5). (i) *The restriction of the sheaf \mathcal{R} to \mathcal{C}^r is a locally free sheaf. Each fiber of the corresponding algebraic vector bundle is a finite dimensional algebra that affords the regular representation of the group W .*

(ii) *For any $s \geq 0$, there is a natural $G \times \mathbb{C}^\times \times \mathbb{C}^\times$ -equivariant isomorphism $\mathcal{R}^{\wedge^s \mathfrak{t}}|_{\mathcal{C}^r} \cong \wedge^s \mathfrak{g}|_{\mathcal{C}^r}$.*

1.6. Small representations. Let L be a finite dimensional rational G -representation. Given a Lie subalgebra $\mathfrak{a} \subset \mathfrak{g}$, we put $L^{\mathfrak{a}} := \{v \in L \mid av = 0, \forall a \in \mathfrak{a}\}$. In particular, we have that $L^{\mathfrak{t}}$ is the zero weight space of L . In §5.1, we introduce a coherent sheaf $L^{\mathfrak{g}}$ on $\mathcal{C}_{\text{norm}}$ such that the geometric fiber of $L^{\mathfrak{g}}$ at any sufficiently general point $(x, y) \in \mathcal{C}_{\text{norm}}$ is the vector space $L^{\mathfrak{g}_{x,y}}$.

Following A. Broer [Br], we call L *small* if the set of weights of L is contained in the root lattice of \mathfrak{g} and, moreover, 2α is not a weight of L , for any root α . Part (i) of our next theorem provides a description of W -isotypic components of the sheaf \mathcal{R} which correspond to the W -representation in the zero weight space of a small G -representation.

Theorem 1.6.1. *Let E be a W -representation, resp. L be a small G -representation. Then, we have*

(i) *There is a canonical isomorphism $L^{\mathfrak{g}} \cong \mathcal{R}^{L^{\mathfrak{t}}}$, of $G \times \mathbb{C}^\times \times \mathbb{C}^\times$ -equivariant $\mathcal{O}_{\mathcal{C}_{\text{norm}}}$ -sheaves.*

(ii) *Restriction to $\mathfrak{T} \subset \mathcal{C}_{\text{norm}}$ induces bigraded $\mathbb{C}[\mathfrak{T}]^W$ -module isomorphisms:*

$$\Gamma(\mathcal{C}_{\text{norm}}, \mathcal{R}^E)^G \xrightarrow{\sim} (E \otimes \mathbb{C}[\mathfrak{T}])^W, \quad \text{resp.} \quad (L \otimes \mathbb{C}[\mathcal{C}_{\text{norm}}])^G \xrightarrow{\sim} (L^{\mathfrak{t}} \otimes \mathbb{C}[\mathfrak{T}])^W.$$

(iii) *For any $s \geq 0$, one has an isomorphism $\Gamma(\mathcal{C}^r, \wedge^s \mathfrak{g})^G \cong (\wedge^s \mathfrak{t} \otimes \mathbb{C}[\mathfrak{T}])^W$.*

Part (i) of the theorem will be proved in §6.5 and parts (ii)-(iii) will be proved in §6.7.

Remark 1.6.2. (a) The adjoint representation $L = \mathfrak{g}$ is small. In this case, one has $L^{\mathfrak{g}} = \mathfrak{g}$. So, the above theorem yields a sheaf isomorphism $\mathfrak{g} \cong \mathcal{R}^{\mathfrak{t}}$ and a bigraded $\mathbb{C}[\mathfrak{T}]^W$ -module isomorphism $(\mathfrak{g} \otimes \mathbb{C}[\mathcal{C}_{\text{norm}}])^G \cong (\mathfrak{t} \otimes \mathbb{C}[\mathfrak{T}])^W$.

(b) Let $G = PGL(V)$ and let $n = \dim V$. The natural $GL(V)$ -action on $(V^*)^{\otimes n} \otimes \wedge^n V$ descends to G and the resulting G -representation L is known to be small. Furthermore, the zero weight space of that representation is isomorphic, as a W -module, to the regular representation of the Symmetric group $W = S_n$. Hence, for $L = (V^*)^{\otimes n} \otimes \wedge^n V$, we have $\mathcal{R} \cong \mathcal{R}^{L^{\mathfrak{t}}}$. Therefore, Theorem 1.6.1(i) yields an isomorphism $\mathcal{R} \cong ((V^*)^{\otimes n} \otimes \wedge^n V)^{\mathfrak{g}}$.

Using the tautological injective morphism $u : L^{\mathfrak{g}} \hookrightarrow L \otimes \mathcal{O}_{\mathcal{C}_{\text{norm}}}$ and writing u^* for the dual morphism one obtains the following *selfdual* diagram of sheaves on $\mathcal{C}_{\text{norm}}$:

$$V^{\otimes n} \otimes \wedge^n V^* \otimes \mathcal{K}_{\mathcal{C}_{\text{norm}}} \xrightarrow{u^*} \mathcal{H}om_{\mathcal{O}_{\mathcal{C}_{\text{norm}}}}(\mathcal{R}, \mathcal{K}_{\mathcal{C}_{\text{norm}}}) \cong \mathcal{R} \xrightarrow{u} (V^*)^{\otimes n} \otimes \wedge^n V \otimes \mathcal{O}_{\mathcal{C}_{\text{norm}}},$$

where the isomorphism in the middle is due to Corollary 1.5.1(ii).

The above diagram is closely related to formula (40) in [Ha1, Proposition 3.7.2].

1.7. Principal nilpotent pairs. Given a regular point $\mathbf{x} = (x_1, x_2) \in \mathcal{C}$, let $\mathcal{R}_{\mathbf{x}}$ be the fiber at \mathbf{x} of (the algebraic vector bundle on \mathcal{C}^r corresponding to) the locally free sheaf \mathcal{R} . By definition, one has $\mathcal{R}_{\mathbf{x}} = \mathbb{C}[p_{\text{norm}}^{-1}(\mathbf{x})]$ where $p_{\text{norm}}^{-1}(\mathbf{x})$, the scheme theoretic fiber of the morphism p_{norm} over \mathbf{x} , is a W -stable (not necessarily reduced) finite subscheme of $\mathfrak{X}_{\text{norm}}$. Thus, $\mathcal{R}_{\mathbf{x}}$ is a finite dimensional algebra equipped with a W -action.

The W -module $\mathcal{R}_{\mathbf{x}}$ is isomorphic to the regular representation of W , by Theorem 1.5.2(i). In particular, one has $\dim \mathcal{R}_{\mathbf{x}}^W = \dim \mathcal{R}_{\mathbf{x}}^{\text{sign}} = 1$. The line $\mathcal{R}_{\mathbf{x}}^W$ is clearly spanned by the unit of the algebra $\mathcal{R}_{\mathbf{x}}$. Further, one has a canonical map $\mathcal{R}_{\mathbf{x}} \rightarrow \mathcal{R}_{\mathbf{x}}^{\text{sign}}$, $r \mapsto r^{\text{sign}}$, the W -equivariant projection to the isotypic component of the sign representation. This map gives, thanks to the isomorphism $\mathcal{K}_{\mathcal{C}_{\text{norm}}} \cong \mathcal{R}^{\text{sign}}$ of Corollary 1.5.1(i), a nondegenerate trace on the algebra

\mathcal{R}_x . In other words, the assignment $r_1 \times r_2 \mapsto (r_1 \cdot r_2)^{\text{sign}}$ provides a nondegenerate symmetric bilinear pairing on \mathcal{R}_x .

The most interesting fibers of the sheaf \mathcal{R} are, in a sense, the fibers over *principal nilpotent pairs*. Following [Gi], we call a regular pair $\mathbf{e} = (e_1, e_2) \in \mathcal{C}^r$ a principal nilpotent pair if there exists a rational homomorphism $\mathbb{C}^\times \times \mathbb{C}^\times \rightarrow G$, $(\tau_1, \tau_2) \mapsto g(\tau_1, \tau_2)$ such that one has

$$\tau_i \cdot e_i = \text{Ad } g(\tau_1, \tau_2)(e_i) \quad i = 1, 2, \quad \forall \tau_1, \tau_2 \in \mathbb{C}^\times. \quad (1.7.1)$$

Given a principal nilpotent pair \mathbf{e} , we introduce a ‘twisted’ $\mathbb{C}^\times \times \mathbb{C}^\times$ -action on \mathfrak{G} given, for $(\tau_1, \tau_2) \in \mathbb{C}^\times \times \mathbb{C}^\times$, by the formula

$$(x, y) \mapsto (\tau_1, \tau_2) \bullet_{\mathbf{e}} (x, y) = (\tau_1 \cdot \text{Ad } g(\tau_1, \tau_2)^{-1}(x), \tau_2 \cdot \text{Ad } g(\tau_1, \tau_2)^{-1}(y)).$$

The $\bullet_{\mathbf{e}}$ -action on \mathfrak{G} , combined with the usual $\mathbb{C}^\times \times \mathbb{C}^\times$ -action on \mathfrak{T} by dilations of the two factors \mathfrak{t} , gives a $\mathbb{C}^\times \times \mathbb{C}^\times$ -action on $\mathfrak{G} \times \mathfrak{T}$. The subvarieties \mathcal{C} and \mathfrak{X} are clearly $\bullet_{\mathbf{e}}$ -stable. Lifting the actions to normalizations, one gets a $\mathbb{C}^\times \times \mathbb{C}^\times$ -action (to be referred to as a $\bullet_{\mathbf{e}}$ -action again) on $\mathcal{C}_{\text{norm}}$, resp. on $\mathfrak{X}_{\text{norm}}$. The map $p_{\text{norm}} : \mathfrak{X}_{\text{norm}} \rightarrow \mathcal{C}_{\text{norm}}$ is $\bullet_{\mathbf{e}}$ -equivariant.

Equations (1.7.1) force e_1, e_2 be nilpotent elements of \mathfrak{g} . The scheme $p_{\text{norm}}^{-1}(\mathbf{e})$ is nonreduced and it has a single closed point, the element $(\mathbf{e}, 0) \in \mathfrak{G} \times \mathfrak{T}$. The point $\mathbf{e} \in \mathcal{C}^r$ is, by construction, a fixed point of the $\bullet_{\mathbf{e}}$ -action on $\mathcal{C}_{\text{norm}}$. Hence, $p_{\text{norm}}^{-1}(\mathbf{e})$ is a $\bullet_{\mathbf{e}}$ -stable subscheme of $\mathfrak{X}_{\text{norm}}$. We conclude that the coordinate ring $\mathcal{R}_{\mathbf{e}} = \mathbb{C}[p_{\text{norm}}^{-1}(\mathbf{e})]$ is a local algebra and the $\mathbb{C}^\times \times \mathbb{C}^\times$ -action on $p_{\text{norm}}^{-1}(\mathbf{e})$ gives a \mathbb{Z}^2 -grading $\mathcal{R}_{\mathbf{e}} = \bigoplus_{m,n \in \mathbb{Z}} \mathcal{R}_{\mathbf{e}}^{m,n}$ on that algebra.

Associated with the nilpotent pair $\mathbf{e} = (e_1, e_2)$ there is a pair of commuting semisimple elements of \mathfrak{g} defined by $h_s := \frac{\partial g(\tau_1, \tau_2)}{\partial \tau_s} \Big|_{\tau_1 = \tau_2 = 1}$, $s = 1, 2$. Let $\mathfrak{g}^1 = \mathfrak{g}_{h_2}$, resp. $\mathfrak{g}^2 = \mathfrak{g}_{h_1}$ (note that the indices are “flipped”). Thus, \mathfrak{g}^s , $s = 1, 2$, is a Levi subalgebra of \mathfrak{g} such that $e_s \in \mathfrak{g}^s$. The Lie algebra $\mathfrak{g}_{h_1, h_2} = \mathfrak{g}^1 \cap \mathfrak{g}^2$ is known to be a Cartan subalgebra of \mathfrak{g} . So, we may (and will) put $\mathfrak{t} := \mathfrak{g}_{h_1, h_2}$. Let R_s^+ be the set of positive roots, resp. W_s the Weyl group, of the reductive Lie algebra \mathfrak{g}^s .

The pair $\mathbf{h} = (h_1, h_2)$ is regular, [Gi, Theorem 1.2]; furthermore, the fiber $p_{\text{norm}}^{-1}(\mathbf{h})$ is a reduced finite subscheme of $\mathfrak{G} \times \mathfrak{T}$. Specifically, writing $W \cdot \mathbf{h}$ for the W -orbit of the element $\mathbf{h} \in \mathfrak{T}$, one has a bijection $W \cdot \mathbf{h} \xrightarrow{\sim} p_{\text{norm}}^{-1}(\mathbf{h})$, $w(\mathbf{h}) \mapsto (\mathbf{h}, w(\mathbf{h}))$. Thus, the algebra $\mathcal{R}_{\mathbf{h}} = \mathbb{C}[p_{\text{norm}}^{-1}(\mathbf{h})]$ is a semisimple algebra isomorphic to $\mathbb{C}[W \cdot \mathbf{h}]$, the coordinate ring of the set $W \cdot \mathbf{h}$.

Let $\mathbb{C}^{\leq m}[\mathfrak{t}]$, $m = 0, 1, 2, \dots$, be the space of polynomials on \mathfrak{t} of degree $\leq m$. We introduce a pair of ascending filtrations on the algebra $\mathbb{C}[\mathfrak{T}] = \mathbb{C}[\mathfrak{t}] \otimes \mathbb{C}[\mathfrak{t}]$ defined by $'F_m \mathbb{C}[\mathfrak{T}] = \mathbb{C}^{\leq m}[\mathfrak{t}] \otimes \mathbb{C}[\mathfrak{t}]$, resp. $''F_m \mathbb{C}[\mathfrak{T}] = \mathbb{C}[\mathfrak{t}] \otimes \mathbb{C}^{\leq m}[\mathfrak{t}]$. The algebra $\mathbb{C}[W \cdot \mathbf{h}]$ is a quotient of the algebra $\mathbb{C}[\mathfrak{T}]$ hence it inherits from $\mathbb{C}[\mathfrak{T}]$ a pair of quotient filtrations $'F \cdot \mathbb{C}[W \cdot \mathbf{h}]$ and $''F \cdot \mathbb{C}[W \cdot \mathbf{h}]$, respectively. We further define bifiltrations

$$F_{m,n} \mathbb{C}[\mathfrak{T}] := 'F_m \mathbb{C}[\mathfrak{T}] \cap ''F_n \mathbb{C}[\mathfrak{T}], \quad F_{m,n} \mathbb{C}[W \cdot \mathbf{h}] := 'F_m \mathbb{C}[W \cdot \mathbf{h}] \cap ''F_n \mathbb{C}[W \cdot \mathbf{h}], \quad m, n \geq 0,$$

on the algebras $\mathbb{C}[\mathfrak{T}]$ and $\mathbb{C}[W \cdot \mathbf{h}]$, respectively. Let $\text{gr}^F \mathbb{C}[\mathfrak{T}]$, resp. $\text{gr}^F \mathbb{C}[W \cdot \mathbf{h}]$, be an associated bigraded algebra, see §7 for more details.

One of the central results of the paper is the following theorem motivated, in part, by [Ha3, §4.1]. Part (i) of the theorem describes how $\mathbb{C}[W \cdot \mathbf{h}] = \mathcal{R}_{\mathbf{h}}$, a semisimple Gorenstein algebra, degenerates to the bigraded Gorenstein algebra $\mathcal{R}_{\mathbf{e}}$.

Theorem 1.7.2. (i) *There is a W -equivariant \mathbb{Z}^2 -graded algebra isomorphism $\mathcal{R}_{\mathbf{e}} \cong \text{gr}^F \mathbb{C}[W \cdot \mathbf{h}]$.*

(ii) *We have $\mathcal{R}_{\mathbf{e}}^{i,j} = 0$ unless $0 \leq i \leq \mathbf{d}_1$ & $0 \leq j \leq \mathbf{d}_2$, where $\mathbf{d}_s := \#R_s^+$, $s = 1, 2$.*

The proof of this theorem occupies §§7.2-7.4. The main idea of the proof is to use the semisimple pair \mathbf{h} to produce a 2-parameter deformation of the point \mathbf{e} inside \mathcal{C}^r . One

may pull-back the sheaf $\mathcal{R}|_{\mathbb{C}^2}$ to the parameter space of the deformation. This way, we obtain a locally free sheaf on \mathbb{C}^2 . On the other hand, a construction based on Rees algebras gives another locally free sheaf on \mathbb{C}^2 such that its fiber over the origin is the vector space $\text{gr}^F \mathbb{C}[W \cdot \mathbf{h}]$. Using a careful analysis (Lemma 7.2.1) based on the theory of W -invariant polynomials on \mathfrak{t} we obtain an isomorphism between the restrictions of the two sheaves in question to the complement of the origin in \mathbb{C}^2 (Proposition 7.3.7). We then exploit the fact that an isomorphism between the restrictions to the punctured plane of two *locally free* sheaves automatically extends across the puncture.

From Theorem 1.7.2 we deduce, cf. §7.6,

Corollary 1.7.3. *There is a natural W -module isomorphism (cf. [Ha3, Proposition 4.1.2]):*

$$\bigoplus_{m \geq 0} \mathcal{R}_{\mathbf{e}}^{0,m} \cong \mathbb{C}[W/W_1], \quad \text{resp.} \quad \bigoplus_{n \geq 0} \mathcal{R}_{\mathbf{e}}^{n,0} \cong \mathbb{C}[W/W_2].$$

One has the following criteria for the variety \mathfrak{X} be normal (hence, by Theorem 1.3.4, also Cohen-Macaulay and Gorenstein) at the point $(\mathbf{e}, 0)$, see §7.6:

Corollary 1.7.4. *The following properties of a principal nilpotent pair \mathbf{e} are equivalent:*

- (i) *The restriction map $\mathbb{C}[\mathfrak{X}] = \Gamma(\mathfrak{C}_{\text{norm}}, \mathcal{R})^G \rightarrow \mathcal{R}_{\mathbf{e}}$ is surjective;*
- (ii) *The natural projection $\text{gr}^F \mathbb{C}[\mathfrak{X}] \rightarrow \text{gr}^F \mathbb{C}[W \cdot \mathbf{h}]$ is surjective;*
- (iii) *The variety \mathfrak{X} is normal at the point $(\mathbf{e}, 0)$.*

In the special case of the group $G = GL_n$, Theorem 1.7.2 follows from the work of M. Haiman, [Ha3, §4.1]. Moreover, in this case, thanks to Haiman's result on the normality of the isospectral Hilbert scheme ([Ha1, Proposition 3.8.4] and to the classification of principal nilpotent pairs in the Lie algebra \mathfrak{gl}_n (see [Gi, Theorem 5.6]), one knows that the equivalent properties of Corollary 1.7.4 hold true for *any* principal nilpotent pair. We remark also that Haiman shows that the validity of the Cohen-Macaulay property of the scheme \mathfrak{X} at each principal nilpotent pair in the Lie algebra \mathfrak{gl}_n is actually equivalent to the validity of the $n!$ -theorem, see [Ha1, Proposition 3.7.3].

On the other hand, Haiman produced an example, see [Ha3, §7.2.1], of a principal nilpotent pair \mathbf{e} for the Lie algebra $\mathfrak{g} = \mathfrak{sp}_6$ where the analogue of the $n!$ -theorem fails, hence the corresponding homomorphism $\text{gr} \mathbb{C}[\mathfrak{X}] \rightarrow \text{gr} \mathbb{C}[W \cdot \mathbf{h}]$ is not surjective and the scheme \mathfrak{X} is not normal at the point $(\mathbf{e}, 0)$. Another example is provided by the *exceptional* principal nilpotent pair in the simple Lie algebra of type E_7 discussed below.

1.8. The polynomial $\Delta_{\mathbf{e}}$. From the isomorphism $\mathcal{R}_{\mathbf{e}} = \text{gr}^F \mathbb{C}[W \cdot \mathbf{h}]$, of Theorem 1.7.2, we see that $\text{gr}^F \mathbb{C}[W \cdot \mathbf{h}]$ is a Gorenstein algebra and that the line $(\text{gr}^F \mathbb{C}[W \cdot \mathbf{h}])^{\text{sign}}$ is the socle of that algebra. In most cases, one can actually obtain a more explicit description of the socle.

To explain the meaning of the words "most cases", we recall that the simple Lie algebra of type E_7 has one 'exceptional' conjugacy class of principal nilpotent pairs $\mathbf{e} = (e_1, e_2)$ such that each of the nilpotent elements e_1 and e_2 has Dynkin labels $\begin{bmatrix} 0 \\ 101010 \end{bmatrix}$. Following [Gi, Definition 4.1], we call a principal nilpotent pair $\mathbf{e} = (e_1, e_2)$, of an arbitrary reductive Lie algebra \mathfrak{g} , *non-exceptional* provided none of the components of \mathbf{e} corresponding to the simple factors of \mathfrak{g} of type E_7 belong to the exceptional conjugacy class of principal nilpotent pairs.

Let $\check{h}_s = \langle h_s, - \rangle$, $s = 1, 2$, be a linear function on \mathfrak{t} that corresponds to the element h_s via the invariant form.

The following result provides a simple description of the socle of the algebra $\text{gr}^F \mathbb{C}[W \cdot \mathbf{h}]$ in the case of nonexceptional nilpotent pairs.

Theorem 1.8.1 (see §7.5). *For any non-exceptional principal nilpotent pair \mathbf{e} , we have*

$$(\text{gr}^F \mathbb{C}[W \cdot \mathbf{h}])^{\text{sign}} = \text{gr}_{\mathbf{d}_1, \mathbf{d}_2} \mathbb{C}[W \cdot \mathbf{h}].$$

Moreover, a base vector of the 1-dimensional vector space on the right is provided by the image under the map $\mathrm{gr}^F \mathbb{C}[\mathfrak{T}] \rightarrow \mathrm{gr}^F \mathbb{C}[W \cdot \mathbf{h}]$ of the class of the following bihomogeneous polynomial:

$$\Delta_{\mathbf{e}} := \sum_{w \in W} \mathrm{sign}(w) \cdot w(\check{h}_1^{\mathbf{d}_1} \otimes \check{h}_2^{\mathbf{d}_2}) \in \mathbb{C}[\mathfrak{T}]^{\mathrm{sign}}. \quad (1.8.2)$$

Remark 1.8.3. The polynomial $\Delta_{\mathbf{e}}$ was first introduced in [Gi]. It is a W -harmonic polynomial on \mathfrak{T} that provides a natural generalization to the case of arbitrary reductive Lie algebras of the Garsia-Haiman polynomial on $\mathbb{C}^n \times \mathbb{C}^n$. The proof of Theorem 1.8.1 is based on the properties of $\Delta_{\mathbf{e}}$ established in [Gi, §4]. The relevance of the notion of non-exceptional pair is due to [Gi, Theorem 4.4], one of the main results of *loc cit*, which says that $\Delta_{\mathbf{e}} \neq 0$ holds if and only if the principal nilpotent pair \mathbf{e} is non-exceptional. \diamond

One may pull-back the function $\Delta_{\mathbf{e}}$ via the projection $\mathfrak{X}_{\mathrm{norm}} \rightarrow \mathfrak{T}$. The resulting W -alternating function on $\mathfrak{X}_{\mathrm{norm}}$ gives a G -invariant section, $\mathbf{s}_{\mathbf{e}}$, of the sheaf $\mathcal{R}^{\mathrm{sign}}$. Let $\mathbf{s}_{\mathbf{e}}(\mathbf{e}) \in \mathcal{R}_{\mathbf{e}}^{\mathrm{sign}}$ denote the value of the section $\mathbf{s}_{\mathbf{e}}$ at the point \mathbf{e} .

Corollary 1.8.4 (see §7.6). *For a non-exceptional principal nilpotent pair \mathbf{e} , one has: $\Delta_{\mathbf{e}}(\mathbf{h}) \neq 0$ and $\mathbf{s}_{\mathbf{e}}(\mathbf{e}) \neq 0$. Moreover, we have that $\mathcal{R}_{\mathbf{e}}^{\mathrm{sign}} = \mathcal{R}_{\mathbf{e}}^{\mathbf{d}_1, \mathbf{d}_2} = \mathbb{C} \cdot \mathbf{s}_{\mathbf{e}}(\mathbf{e})$ is the socle of the algebra $\mathcal{R}_{\mathbf{e}}$.*

Let $\langle -, - \rangle : \mathfrak{T} \times \mathfrak{T} \rightarrow \mathbb{C}$ be the natural $W \times W$ -invariant bilinear form given by the formula $\langle \mathbf{x}, \mathbf{y} \rangle := \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle$, for any $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2) \in \mathfrak{T}$.

We define a holomorphic function \mathbf{E} on $\mathfrak{T} \times \mathfrak{T}$ as follows

$$\mathbf{E}(\mathbf{x}, \mathbf{y}) := \sum_{w \in W} \mathrm{sign}(w) \cdot e^{\langle \mathbf{x}, w(\mathbf{y}) \rangle}, \quad \mathbf{x}, \mathbf{y} \in \mathfrak{T}.$$

It is clear that, for any \mathbf{x}, \mathbf{y} , we have $\mathbf{E}(\mathbf{x}, \mathbf{y}) = \mathbf{E}(\mathbf{y}, \mathbf{x})$. Observe also that the function \mathbf{E} is W -invariant with respect to the diagonal action on $\mathfrak{T} \times \mathfrak{T}$ and, for any fixed $\mathbf{x} \in \mathfrak{G}$, the function $\mathbf{E}(\mathbf{x}, -) = \mathbf{E}(-, \mathbf{x})$ is a W -alternating holomorphic function on \mathfrak{T} .

Similarly to the above, one can pull-back the function \mathbf{E} via the projection $\mathfrak{X}_{\mathrm{norm}} \times \mathfrak{X}_{\mathrm{norm}} \rightarrow \mathfrak{T} \times \mathfrak{T}$. The resulting function on $\mathfrak{X}_{\mathrm{norm}} \times \mathfrak{X}_{\mathrm{norm}}$ gives a $G \times G$ -invariant holomorphic section, \mathbf{S} , of the coherent sheaf $\mathcal{R}^{\mathrm{sign}} \boxtimes \mathcal{R}^{\mathrm{sign}}$ on $\mathfrak{C} \times \mathfrak{C}$.

The following result shows that the section \mathbf{S} provides a canonical holomorphic interpolation between the algebraic sections $\mathbf{s}_{\mathbf{e}}$ associated with various, not necessarily G -conjugate, principal nilpotent pairs \mathbf{e} of the Lie algebra \mathfrak{g} .

Proposition 1.8.5 (§7.6). *For any non-exceptional principal nilpotent pair \mathbf{e} , in $\mathcal{R}^{\mathrm{sign}} \boxtimes \mathcal{R}_{\mathbf{e}}^{\mathrm{sign}}$, resp. in $\mathcal{R}_{\mathbf{e}}^{\mathrm{sign}} \boxtimes \mathcal{R}^{\mathrm{sign}}$, one has*

$$\mathbf{S}|_{\mathfrak{C} \times \{\mathbf{e}\}} = c \cdot \mathbf{s}_{\mathbf{e}} \boxtimes \mathbf{s}_{\mathbf{e}}(\mathbf{e}), \text{ resp. } \mathbf{S}|_{\{\mathbf{e}\} \times \mathfrak{C}} = c \cdot \mathbf{s}_{\mathbf{e}}(\mathbf{e}) \boxtimes \mathbf{s}_{\mathbf{e}} \text{ where } c = \frac{1}{\mathbf{d}_1! \cdot \mathbf{d}_2! \cdot \Delta_{\mathbf{e}}(\mathbf{h})}.$$

1.9. Relation to work of M. Haiman. Let $\mathrm{Hilb}^n(\mathbb{C}^2)$ be the Hilbert scheme of n points in \mathbb{C}^2 . In his work on the $n!$ -theorem, Haiman introduced a certain isospectral Hilbert scheme $\widetilde{\mathrm{Hilb}}^n(\mathbb{C}^2)$, a reduced finite scheme over $\mathrm{Hilb}^n(\mathbb{C}^2)$, see [Ha1]. The main result of *loc cit* says that $\widetilde{\mathrm{Hilb}}^n(\mathbb{C}^2)$ is a normal, Cohen-Macaulay and Gorenstein scheme.

Now, let $G = GL_n$. It turns out that there is a Zariski open dense subset $\mathfrak{C}^\circ \subset \mathfrak{C}^r$ such that the projection $p_{\mathrm{norm}} : p_{\mathrm{norm}}^{-1}(\mathfrak{C}^\circ) \rightarrow \mathfrak{C}^\circ$ is closely related to the projection $\widetilde{\mathrm{Hilb}}^n(\mathbb{C}^2) \rightarrow \mathrm{Hilb}^n(\mathbb{C}^2)$, see §8. Using this relation, we are able to deduce from our Theorem 1.3.4 that the normalization of the isospectral Hilbert scheme is Cohen-Macaulay and Gorenstein, see Proposition 8.2.4. Unfortunately, our approach does not seem to yield an independent proof of normality of the isospectral Hilbert scheme, while the proof of normality given in [Ha1,

Proposition 3.8.4] is based on the ‘polygraph theorem’ [Ha1, Theorem 4.1], a key technical result of [Ha1].

Nonetheless, we are able to use the locally free sheaf $\mathcal{R}|_{\mathcal{C}^r}$ to construct a rank $n!$ algebraic vector bundle \mathcal{P} on $\text{Hilb}^n(\mathbb{C}^2)$ whose fibers afford the regular representation of the Symmetric group. The results of Haiman [Ha1] then insure that the vector bundle \mathcal{P} is isomorphic, *a posteriori*, to the *Procesi bundle*, cf. Corollary 8.2.5.

We note that the properties of the vector bundle \mathcal{P} that we construct are often sufficient (without the knowledge of the isomorphism with the Procesi bundle) for applications. This is so, for instance, in the new proof of the positivity of Kostka-Macdonald polynomials found recently by I. Gordon [Go]. Another situation of this kind is provided by Theorem 1.9.1 below. See also various combinatorial results established in [BG].

Let $V = \mathbb{C}^n$ be the fundamental representation of $G = GL_n$ and let \mathcal{V} be the tautological rank n vector bundle on $\text{Hilb}^n(\mathbb{C}^2)$, see §8.3.

Given an integer $m \geq 0$, let $\mathbb{C}^m[V]$ be the space of degree m homogeneous polynomials on V , and let S_m be the symmetric group on m letters. The group S_m acts naturally on $V^{\otimes m}$, resp. on $\mathcal{V}^{\otimes m}$, hence also on $\mathbb{C}[\mathfrak{X}_{\text{norm}}] \otimes \mathbb{C}^m[V] \otimes V^{\otimes m}$, resp. on $\mathcal{P} \otimes \mathcal{V}^{\otimes m}$, via the action on the last tensor factor.

We also have the group $W = S_n$ and a natural $W \times \mathbb{C}^\times \times \mathbb{C}^\times$ action on $\mathfrak{X}_{\text{norm}}$, resp. on \mathcal{P} , hence on $\mathbb{C}[\mathfrak{X}_{\text{norm}}] \otimes \mathbb{C}^m[V] \otimes V^{\otimes m}$, resp. on $\mathcal{P} \otimes \mathcal{V}^{\otimes m}$, via the action on the first factor. Finally, we let the subgroup $SL_n \subset GL_n$ act diagonally on $\mathbb{C}[\mathfrak{X}_{\text{norm}}] \otimes \mathbb{C}^m[V] \otimes V^{\otimes m}$. The SL_n -action commutes with the previously defined actions of other groups. Thus, the space $(\mathbb{C}[\mathfrak{X}_{\text{norm}}] \otimes \mathbb{C}^m[V] \otimes V^{\otimes m})^{SL_n}$, of SL_n -invariants, acquires the natural structure of an $S_m \times W$ -equivariant bigraded $\mathbb{C}[\mathfrak{T}]$ -module.

Theorem 1.9.1 (see §8.3). *There is a $W \times S_m$ -equivariant bigraded $\mathbb{C}[\mathfrak{T}]$ -module isomorphism*

$$(\mathbb{C}[\mathfrak{X}_{\text{norm}}] \otimes \mathbb{C}^m[V] \otimes V^{\otimes m})^{SL_n} \xrightarrow{\sim} \Gamma(\text{Hilb}^n(\mathbb{C}^2), \mathcal{P} \otimes \mathcal{V}^{\otimes m}), \quad \forall m \geq 0.$$

We remark that the space of global sections that appears on the right hand side of the isomorphism above is the “polygraph space” that played a key role in the work of Haiman. In [Ha4, Theorem 3.1], Haiman gives a bigraded character formula for the polygraph space in terms of Macdonald polynomials. On the other hand, our Theorem 1.4.1 may be used to find the bigraded character of the vector space $(\mathbb{C}[\mathfrak{X}_{\text{norm}}] \otimes \mathbb{C}^m[V] \otimes V^{\otimes m})^{SL_n}$. Combining these results with Theorem 1.9.1, one obtains a certain explicit identity that relates the formal power series in the right hand side of the formula of Theorem 1.4.1 to Macdonald polynomials, see [BG] for more details.

A particularly nice special case occurs in the situation where the integer m , in Theorem 1.9.1, is a multiple of n , i.e., such that $m = kn$ for some integer $k \geq 1$. Let $A^k \subset \mathbb{C}[\mathfrak{T}]$ be the vector space spanned by the products of k -tuples of elements of $\mathbb{C}[\mathfrak{T}]^{\text{sign}}$, the space of W -alternating polynomials. The bigrading on $\mathbb{C}[\mathfrak{T}]$ induces one on A^k and we have

Proposition 1.9.2. *For any $k \geq 1$, there is a natural bigraded isomorphism (see §8.5):*

$$(\mathbb{C}[\mathfrak{C}_{\text{red}}] \otimes \mathbb{C}^{kn}[V])^{SL_n} \xrightarrow{\sim} A^k.$$

This result may be viewed as an example of a situation involving the isomorphism of Theorem 1.9.1 where taking normalizations is unnecessary.

1.10. Layout of the paper. In section 2 we begin with basic properties of commuting and isospectral varieties. We then review the material involving holonomic \mathcal{D} -modules and (polarizable) Hodge modules that will be used in the paper. We show that the Harish-Chandra

module \mathcal{M} is a simple holonomic \mathcal{D} -module and give \mathcal{M} the structure of a Hodge module. We explain that Saito's results on Hodge modules imply easily that the coherent sheaf $\mathrm{gr}^{\mathrm{Hodge}} \mathcal{M}$ is Cohen-Macaulay and Gorenstein. This allows us to reduce the proof of Theorem 1.3.3 to an isomorphism $\mathrm{gr}^{\mathrm{Hodge}} \mathcal{M}|_{\mathfrak{X}^{rr}} \cong \mathcal{O}_{\mathfrak{X}^{rr}}$, where \mathfrak{X}^{rr} is a certain open subset of \mathfrak{X} , and to a codimension estimate involving \mathfrak{X}^{rr} . We finish the section by showing how Theorem 1.3.3 implies Theorem 1.3.4.

In section 3, we recall the definition of the Grothendieck-Springer resolution and construct its 'double analogue'. We introduce a DG algebra \mathcal{A} such that its homogeneous components are locally free sheaves on the double analogue of the Grothendieck-Springer resolution. The spectrum of the DG algebra \mathcal{A} may be thought of as a DG resolution of the isospectral commuting variety.

In section 4, we recall the Hotta-Kashiwara construction of the Harish-Chandra module \mathcal{M} as a direct image of the structure sheaf of the Grothendieck-Springer resolution. We use the Hotta-Kashiwara construction to obtain a description of the sheaf $\mathrm{gr}^{\mathrm{Hodge}} \mathcal{M}$ in terms of the DG algebra \mathcal{A} . From that description, we deduce the above mentioned isomorphism $\mathrm{gr}^{\mathrm{Hodge}} \mathcal{M}|_{\mathfrak{X}^{rr}} \cong \mathcal{O}_{\mathfrak{X}^{rr}}$.

In section 5, we give the definition of the 'universal stabilizer' sheaf. Further, we associate with any finite dimensional representation L of the Lie algebra \mathfrak{g} a coherent sheaf $L^{\mathfrak{g}}$, on \mathfrak{g}^r . In the case where the representation L is small, we obtain, using a refinement of an idea due to Beilinson and Kazhdan, a description of the sheaf $L^{\mathfrak{g}}$ in terms the 'universal zero weight space' of L . At the end of the section we indicate how to modify the construction in order to obtain a similar description of $L^{\mathfrak{g}}$ for a not necessarily small representation L .

In section 6, we introduce a stratification of the commuting variety and use it to prove the above mentioned dimension estimate involving the set \mathfrak{X}^{rr} which is required for our proof of Theorem 1.3.3. We then prove Theorems 1.5.2 and 1.6.1 by adapting the construction of §5 to a 'double' setting. We also prove Corollary 1.5.1.

Section 7 is devoted to the proofs of all the results stated in §§1.7-1.8 of the Introduction.

In section 8, we consider the case of the group GL_n . We apply our results about commuting varieties to the geometry of the isospectral Hilbert scheme. We discuss relations with the work of M. Haiman and prove the results stated in §1.9.

Finally, in section 9, we prove Corollary 1.3.6 and Corollary 1.3.7.

We remark that the proof of the main results, Theorem 1.3.3 and Theorem 1.3.4, is rather long. The argument involves several ingredients of different nature. The discussion of these ingredients is spread over a number of sections of the paper. Our division into sections has been made according to the nature of the material rather than to the order in which that material is actually used in the proof.

To make the logical structure of our arguments more clear, below is an outline of the main steps of the proof of Theorem 1.3.3 and Theorem 1.3.4, listed in the *logical* order:

- §§2.6, 6.3: Define a Zariski open smooth subset $\mathfrak{X}^{rr} \subset \mathfrak{X}$ and show that the set $\mathfrak{X} \setminus \mathfrak{X}^{rr}$ has codimension ≥ 2 in \mathfrak{X} (Corollary 2.6.5).
- §§3.1, 3.3: Introduce the 'double analogue' of the Grothendieck-Springer resolution and show that it is an isomorphism over \mathfrak{X}^{rr} (Corollary 3.3.5).
- §§2.4, 2.5: Show that \mathcal{M} is a simple \mathcal{D} -module with the natural structure of a Hodge module.
- §§4.2, 4.4: Use the Hotta-Kashiwara construction of \mathcal{M} to get a description (see Corollary 4.4.6) of $\mathrm{gr}^{\mathrm{Hodge}} \mathcal{M}$ in terms of the double analogue of the Grothendieck-Springer resolution.
- §4.5: Establish an isomorphism: $\mathrm{gr}^{\mathrm{Hodge}} \mathcal{M}|_{\mathfrak{X}^{rr}} \cong \mathcal{O}_{\mathfrak{X}^{rr}}$ (see Proposition 2.6.6 and §4.5).
- §2.5: Deduce from Saito's theory that the sheaf $\mathrm{gr}^{\mathrm{Hodge}} \mathcal{M}$ is Cohen-Macaulay and selfdual (Corollary 2.5.1).

§2.6: Deduce Theorem 1.3.3 from Lemma 2.6.1, using Proposition 2.6.6 and Corollary 2.6.5 (which were proved at an earlier stage). Deduce Theorem 1.3.4 from Theorem 1.3.3 and Corollary 2.5.1.

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2. ANALYSIS OF THE HARISH-CHANDRA MODULE

2.1. **Isospectral varieties.** Let $\mathfrak{g}^{rs} \subset \mathfrak{g}^r$ be the set of regular semisimple elements, resp. $\mathfrak{C}^{rs} \subset \mathfrak{C}^r$ be the set of pairs $(x, y) \in \mathfrak{C}^r$ such that both x and y are semisimple.

Basic properties of the commuting scheme \mathfrak{C} may be summarized as follows.

Proposition 2.1.1. (i) *The set \mathfrak{C}^{rs} is Zariski open and dense in \mathfrak{C} . Thus, \mathfrak{C} is a generically reduced and irreducible scheme; moreover, we have $\dim \mathfrak{C} = \dim \mathfrak{g} + r$.*

(ii) *The smooth locus of the scheme \mathfrak{C} equals \mathfrak{C}^r ; for any $(x, y) \in \mathfrak{C}$ one has $\dim \mathfrak{g}_{x,y} \geq r$.*

Here, part (i) is due to Richardson [Ri1]. For the proof of part (ii) see e.g. [Po, Lemma 2.3], or Remark 6.6.2 of section 6 below.

Let $\mathfrak{g} // G := \operatorname{Spec} \mathbb{C}[\mathfrak{g}]^G$, resp. $\mathfrak{C} // G := \operatorname{Spec} \mathbb{C}[\mathfrak{C}]^G$, be the categorical quotient. The natural restriction homomorphism $\mathbb{C}[\mathfrak{g}]^G \rightarrow \mathbb{C}[\mathfrak{t}]^W$, resp. $\mathbb{C}[\mathfrak{C}]^G \rightarrow \mathbb{C}[\mathfrak{T}]^W$, cf. (1.3.1), gives morphisms of schemes $\mathfrak{t} \rightarrow \mathfrak{t}/W \rightarrow \mathfrak{g} // G$, resp. $\mathfrak{T} \rightarrow \mathfrak{T}/W \rightarrow \mathfrak{C} // G$.

Remark 2.1.2. The morphism $\mathfrak{t}/W \rightarrow \mathfrak{g} // G$ is an isomorphism by the Chevalley restriction theorem. One can show, using that the map $\mathbb{C}[\mathfrak{C}]^G \rightarrow \mathbb{C}[\mathfrak{T}]^W$ is surjective by a theorem of Joseph [Jo], that the morphism $\mathfrak{T}/W \rightarrow \mathfrak{C} // G$ induces an isomorphism $\mathfrak{T}/W \xrightarrow{\sim} [\mathfrak{C} // G]_{\text{red}}$. It is expected that the scheme $\mathfrak{C} // G$ is in fact reduced. This is known to be so in the special case of the group $G = GL_n$, see [GG, Theorem 1.3]. \diamond

Next, we form a fiber product $\mathfrak{x} := \mathfrak{g} \times_{\mathfrak{g} // G} \mathfrak{t}$, resp. $\mathfrak{C} \times_{\mathfrak{C} // G} \mathfrak{T}$, a closed $G \times W$ -stable subscheme of $\mathfrak{g} \times \mathfrak{t}$, resp. of $\mathfrak{C} \times \mathfrak{T}$. It is clear that the set $\mathfrak{x}^{rs} := \mathfrak{g}^{rs} \times_{\mathfrak{g} // G} \mathfrak{t}$ is a smooth Zariski open subset of \mathfrak{x} , resp. $\mathfrak{x}^{rs} := \mathfrak{C}^{rs} \times_{\mathfrak{C} // G} \mathfrak{T}$ is a smooth Zariski open subset of $\mathfrak{C} \times_{\mathfrak{C} // G} \mathfrak{T}$. The first projection $\mathfrak{x} \rightarrow \mathfrak{g}$, resp. $\mathfrak{C} \times_{\mathfrak{C} // G} \mathfrak{T} \rightarrow \mathfrak{C}$, is a G -equivariant *finite* morphism. The group W acts along the fibers of this morphism.

Part (i) of the following lemma is Corollary 6.2.3 of §6 below, and part (ii) is clear.

Lemma 2.1.3. (i) *The set \mathfrak{x}^{rs} , resp. \mathfrak{x}^{rs} , is an irreducible dense subset of \mathfrak{x} , resp. of $\mathfrak{C} \times_{\mathfrak{C} // G} \mathfrak{T}$.*

(ii) *The first projection $\mathfrak{x}^{rs} \rightarrow \mathfrak{g}^{rs}$, resp. $\mathfrak{x}^{rs} \twoheadrightarrow \mathfrak{C}^{rs}$, is a Galois covering with Galois group W .*

The scheme \mathfrak{x} is known to be a reduced normal complete intersection in $\mathfrak{g} \times \mathfrak{t}$, cf. eg [BB]. On the contrary, the scheme $\mathfrak{C} \times_{\mathfrak{C} // G} \mathfrak{T}$ is not reduced already in the case $\mathfrak{g} = \mathfrak{sl}_2$. This motivated M. Haiman, [Ha2, §8], [Ha3, §7.2], to introduce the following definition, which is equivalent to formula (1.3.2) of the Introduction.

Definition 2.1.4. The *isospectral commuting variety* is defined as $\mathfrak{X} := [\mathfrak{C} \times_{\mathfrak{C} // G} \mathfrak{T}]_{\text{red}}$, a *reduced* fiber product. Let $p : \mathfrak{X} \rightarrow \mathfrak{C}$, resp. $p_{\mathfrak{T}} : \mathfrak{X} \rightarrow \mathfrak{T}$, denote the first, resp. second, projection.

Lemma 2.1.3(i) shows that \mathfrak{X} is an irreducible variety and that we may (and will) identify the set \mathfrak{X}^{rs} with a smooth Zariski open and dense subset of \mathfrak{X} .

Let $\mathfrak{r} := \{(x, t) \in \mathfrak{r} \mid x \in \mathfrak{g}^r\}$. This is a Zariski open and dense subset of \mathfrak{r} which is contained in the smooth locus of \mathfrak{r} (since the differential of the adjoint quotient map $\mathfrak{g} \rightarrow \mathfrak{g}/G$ is known to have maximal rank at any point of \mathfrak{g}^r). Let $N_{\mathfrak{r}}$ be the total space of the conormal bundle on \mathfrak{r} in $\mathfrak{g} \times \mathfrak{t}$ and let $\overline{N_{\mathfrak{r}}}$ be the closure of $N_{\mathfrak{r}}$ in $T^*(\mathfrak{g} \times \mathfrak{t})$.

In general, let X be a smooth variety. An irreducible reduced subvariety $\Lambda \subset T^*X$ is said to be *Lagrangian* if the tangent space to Λ at any smooth point of Λ is a Lagrangian subspace of the tangent space to T^*X with respect to the canonical symplectic 2-form on T^*X .

Lemma 2.1.5. *In $T^*(\mathfrak{g} \times \mathfrak{t}) = \mathfrak{G} \times \mathfrak{T}$, we have $\overline{N_{\mathfrak{r}}} = \mathfrak{X}$. In particular, \mathfrak{X} is a $\mathbb{C}^\times \times \mathbb{C}^\times$ -stable Lagrangian subvariety and $N_{\mathfrak{r}}$ is a smooth Zariski open and dense subset of \mathfrak{X} .*

Proof. Let $\mathfrak{t}' := \mathfrak{t} \cap \mathfrak{g}^r$ and $\mathfrak{X}' := \{(x, y, t_1, t_2) \in \mathfrak{X} \mid x \in \mathfrak{g}^{rs}\}$. We claim that $\mathfrak{X}' \subset N_{\mathfrak{r}}$. To see this we observe that the assignment $gT \times (t_1, t_2) \mapsto (\text{Ad } g(t_1), \text{Ad } g(t_2), t_1, t_2)$ yields an isomorphism $(G/T) \times \mathfrak{t}' \times \mathfrak{t} \xrightarrow{\sim} \mathfrak{X}'$, see Lemma 6.2.2. Thus, by G -equivariance, it suffices to show that every point of \mathfrak{X}' of the form (t_1, t_2, t_1, t_2) is contained in $N_{\mathfrak{r}}$. Any tangent vector to \mathfrak{r} at the point $(t_1, t_1) \in \mathfrak{g} \times \mathfrak{t}$ has the form $\xi = ([a, t_1] + t, t) \in \mathfrak{g} \times \mathfrak{t}$, for some $a \in \mathfrak{g}$ and some $t \in \mathfrak{t}$. Further, the covector that corresponds to the element (t_1, t_2, t_1, t_2) via our identification $\mathfrak{G} \times \mathfrak{T} \xrightarrow{\sim} T^*\mathfrak{g} \times T^*\mathfrak{t} = T^*(\mathfrak{g} \times \mathfrak{t})$ is the linear function $\xi^* : \mathfrak{g} \times \mathfrak{t} : (u, v) \mapsto \langle t_2, u \rangle - \langle t_2, v \rangle$. Therefore, using the invariance of the bilinear form $\langle -, - \rangle$, we find

$$\langle \xi^*, \xi \rangle = \langle t_2, [a, t_1] + t \rangle - \langle t_2, t \rangle = \langle t_2, [a, t_1] \rangle = \langle a, [t_1, t_2] \rangle = 0.$$

Thus, we have proved that $\mathfrak{X}' \subset N_{\mathfrak{r}}$. Moreover, it is clear that \mathfrak{X}' is an open subset of $\mathfrak{X} \cap N_{\mathfrak{r}}$ and, we have $\dim \mathfrak{X}' = \dim(G/T \times \mathfrak{t}' \times \mathfrak{t}) = \dim(\mathfrak{g} \times \mathfrak{t}) = \frac{1}{2} \dim T^*(\mathfrak{g} \times \mathfrak{t}) = \dim N_{\mathfrak{r}}$. Now, Lemma 2.1.3(i) implies that \mathfrak{X} is reduced and irreducible. Hence, \mathfrak{X}' is a dense subset both in \mathfrak{X} and in $N_{\mathfrak{r}}$. It follows that $\mathfrak{X} = \overline{\mathfrak{X}'} = \overline{N_{\mathfrak{r}}}$. \square

2.2. Let X be a smooth variety and $q : T^*X \rightarrow X$ the cotangent bundle. The sheaf \mathcal{D}_X comes equipped with an ascending filtration $F^\bullet \mathcal{D}_X$ by the order of differential operator. For the associated graded sheaf, one has a canonical isomorphism $\text{gr}^{\text{ord}} \mathcal{D}_X \cong q_* \mathcal{O}_{T^*X}$.

Let M be a \mathcal{D}_X -module. An ascending filtration $F_\bullet M$ such that $F_i^{\text{ord}} \mathcal{D}_X \cdot F_j M \subset F_{i+j} M$, $\forall i, j$, is said to be *good* if $\text{gr}^F M$, an associated graded module, is a coherent $q_* \mathcal{O}_{T^*X}$ -module. In that case, there is a canonically defined coherent sheaf $\widetilde{\text{gr}}^F M$ on T^*X such that one has an isomorphism $\text{gr}^F M = q_* \widetilde{\text{gr}}^F M$ of $q_* \mathcal{O}_{T^*X}$ -modules. We write $[\text{Supp}(\widetilde{\text{gr}}^F M)]$ for the *support cycle* of the sheaf $\widetilde{\text{gr}}^F M$, a linear combination of the irreducible components of the support of $\widetilde{\text{gr}}^F M$ counted with multiplicities. This is an algebraic cycle in T^*X which is known to be independent of the choice of a good filtration on M , cf. [Bo].

Recall that M is called *holonomic* if one has $\dim \text{Supp}(\widetilde{\text{gr}}^F M) = \dim X$. In such a case, each irreducible component of $\text{Supp}(\widetilde{\text{gr}}^F M)$, viewed as a reduced variety, is a Lagrangian subvariety of T^*X . Holonomic \mathcal{D}_X -modules form an abelian category and the assignment $M \mapsto [\text{Supp}(\widetilde{\text{gr}}^F M)]$ is additive on short exact sequences of holonomic modules, cf. eg. [Bo], [HTT]. From this, one obtains

Lemma 2.2.1. *If M is a \mathcal{D}_X -module such that the cycle $[\text{Supp}(\widetilde{\text{gr}}^F M)]$ equals the fundamental cycle of a Lagrangian subvariety, taken with multiplicity 1, then M is a simple holonomic \mathcal{D}_X -module.*

Given a morphism $f : X \rightarrow Y$ of smooth varieties, let $\mathcal{D}_{X \rightarrow Y} := \mathcal{O}_X \otimes_{f^{-1} \mathcal{O}_Y} f^{-1} \mathcal{D}_Y$. Assuming f is proper, there is a direct image functor $\int_f^R = Rf_*(- \otimes_{\mathcal{D}_X}^L \mathcal{D}_{X \rightarrow Y})$ between

bounded derived categories of coherent *right* \mathcal{D} -modules on X and Y , respectively. The corresponding functor on left \mathcal{D} -modules is then defined by $\int_f M = \mathcal{K}_Y^{-1} \otimes_{\mathcal{O}_Y} \int_f^R (\mathcal{K}_X \otimes_{\mathcal{O}_X} M)$, see [HTT, pp.22-23, 69] for more details.

In the special case where $f : X \hookrightarrow Y$ is a closed immersion, one has $\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y = \mathcal{D}_Y|_X$ so, we get $\int_f^R \mathcal{K}_X = f_*(\mathcal{K}_X \otimes_{\mathcal{O}_X} (\mathcal{D}_Y|_X))$. This \mathcal{D}_Y -module comes equipped with a natural order filtration defined by the formula $F_m^{\text{ord}}(\int_f^R \mathcal{K}_X) = f_*(\mathcal{K}_X \otimes_{\mathcal{O}_X} (F_m^{\text{ord}}\mathcal{D}_Y)|_X)$, $m \geq 0$. Thus, writing $q : T^*Y \rightarrow Y$ for the cotangent bundle, in this case one obtains

$$\widetilde{\text{gr}}^{\text{ord}}\left(\int_f^R \mathcal{K}_Y\right) = q^*(f_*\mathcal{K}_X). \quad (2.2.2)$$

2.3. Filtered \mathcal{D} -modules and duality. Below, we will use rudiments of the formalism of filtered derived categories. Let

$$E : \cdots \xrightarrow{d_{k-2}} E^{k-1} \xrightarrow{d_{k-1}} E^k \xrightarrow{d_k} E^{k+1} \xrightarrow{d_{k+1}} \cdots,$$

be a filtered complex in an abelian category. One has the following useful

Definition 2.3.1. The filtered complex E is said to be *strict* if the morphism $d_k : F_j E^k \rightarrow \text{Im } d_k \cap F_j E^{k+1}$ is surjective, for any $k, j \in \mathbb{Z}$.

This notion only depends on the quasi-isomorphism class of E in an appropriate filtered derived category.

Given a filtered complex E , there is an induced filtration on each cohomology group $H^k(E)$, $k \in \mathbb{Z}$. It is immediate from the construction of the standard spectral sequence of a filtered complex that the filtered complex E is strict if and only if the spectral sequence $H^*(\text{gr } E) \Rightarrow \text{gr } H^*(E)$ degenerates at the first page, cf. [La, Lemma 3.3.5]. In that case, one has a canonical isomorphism $\text{gr } H^*(E) \cong H^*(\text{gr } E)$.

Following G. Laumon and M. Saito, for any smooth algebraic variety X , one has an exact (not abelian) category of filtered left \mathcal{D}_X -modules and also the corresponding derived category. Thus, let $\text{FD}_X\text{-mod}$ be an additive category whose objects are \mathcal{D}_X -modules M equipped with a good filtration F . Further, abusing the notation slightly, we let $D_{\text{coh}}^b(\text{FD}_X)$ be the triangulated category whose objects are isomorphic to bounded complexes (M, F) , of filtered \mathcal{D}_X -modules, such that each cohomology group $\mathcal{H}^i(M, F)$ is an object of $\text{FD}_X\text{-mod}$, cf. [La], [Sa].

In his work [Sa], M. Saito defines a semisimple abelian category $\text{HM}(X)$ of polarizable Hodge modules, see [Sa, §5.2.10]. The data of a polarizable Hodge module includes, in particular, a holonomic \mathcal{D}_X -module M with regular singularities and a good filtration F on M called *Hodge filtration*.² Thus, (M, F) is a filtered \mathcal{D}_X -module; abusing notation, we write $(M, F) \in \text{HM}(X)$ and let $\widetilde{\text{gr}}(M, F)$ denote the corresponding coherent sheaf on T^*X .

Let $\mathbb{D}(-) = R\mathcal{H}om_{\mathcal{D}_X}(-, \mathcal{D}_X \otimes_{\mathcal{O}_X} K_X^{-1}[\dim X])$ be the standard Verdier duality functor on \mathcal{D} -modules, cf. [HTT, §2.6]. Laumon and Saito have upgraded Verdier duality to a triangulated contravariant duality functor $\text{F}\mathbb{D}$ on the category $D_{\text{coh}}^b(\text{FD}_X)$, cf. [La, §4], [Sa, §2]. Furthermore, Saito showed that, for any $(M, F) \in \text{HM}(X)$, the filtered complex $\text{F}\mathbb{D}(M, F) \in D_{\text{coh}}^b(\text{FD}_X)$ is strict, see [Sa, Lemma 5.1.13].

Let $\text{Coh } Z$ denote the abelian category of coherent sheaves on a scheme Z and let $D_{\text{coh}}^b(Z)$ denote the corresponding bounded derived category.

²Part of the data involving “polarization” will play no role in the present work. For this reason, from now on, we will refer to polarizable Hodge modules as ‘Hodge modules’, for short.

The assignment $(M, F) \mapsto \widetilde{\text{gr}}^F M$ gives a functor $\text{FD}_X\text{-mod} \rightarrow \text{Coh } T^*X$ that can be extended to a triangulated functor $\widetilde{\text{dgr}} : D_{\text{coh}}^b(\text{FD}_X) \rightarrow D_{\text{coh}}^b(T^*X)$, cf. [La], (4.0.6). It is essentially built into the construction of duality on filtered derived categories that the functor $\widetilde{\text{dgr}}$ intertwines the duality FID on $D_{\text{coh}}^b(\text{FD}_X)$ with the Grothendieck-Serre duality on $D_{\text{coh}}^b(T^*X)$, [La, §4], [Sa, §2].

Now let $(M, F) \in \text{HM}(X)$. Then M is a holonomic \mathcal{D} -module. Hence $\mathbb{D}(M)$, viewed as an object of the derived category, is quasi-isomorphic to its 0th cohomology. Therefore, the strictness of the filtered complex $\text{FID}(M, F)$ implies that the complex $\widetilde{\text{dgr}}(\text{FID}(M, F))$ has nonvanishing cohomology in at most one degree. It follows, in view of the above, that the Grothendieck-Serre dual of the object $\widetilde{\text{gr}}(M, F) \in D_{\text{coh}}^b(T^*X)$ has nonvanishing cohomology in at most one degree again. Moreover, the cohomology sheaf in that degree is isomorphic to $\widetilde{\text{gr}}(\mathbb{D}(M))$. In particular, one concludes that $\widetilde{\text{gr}}(M, F)$ is a Cohen-Macaulay sheaf on T^*X , for any $(M, F) \in \text{HM}(X)$, see [Sa], Lemma 5.1.13.

2.4. The order filtration on the Harish-Chandra module. Observe that $\mathcal{D}(\mathfrak{g}) \cdot \text{ad } \mathfrak{g}$, a left ideal of the algebra $\mathcal{D}(\mathfrak{g})$, is stable under the G -action on $\mathcal{D}(\mathfrak{g})$ induced by the adjoint action on \mathfrak{g} . Multiplication in the algebra $\mathcal{D}(\mathfrak{g})$ gives the quotient $A := \mathcal{D}(\mathfrak{g})^G / [\mathcal{D}(\mathfrak{g}) \cdot \text{ad } \mathfrak{g}]^G$ a natural algebra structure called *quantum Hamiltonian reduction*, cf. eg. [GG, §7.1]. Furthermore, it gives $\mathcal{D}(\mathfrak{g}) / \mathcal{D}(\mathfrak{g}) \cdot \text{ad } \mathfrak{g}$ the natural structure of an $(\mathcal{D}(\mathfrak{g}), A)$ -bimodule, [LS3], [GG, §7.1].

It is immediate from definitions that the radial part map rad considered in §1.2 descends to a well defined map $A \rightarrow \mathcal{D}(\mathfrak{t})^W$. According to an important result, due to Levasseur and Stafford [LS1]-[LS2] and Wallach [Wa], the latter map is an algebra isomorphism. Thus, one may view $\mathcal{D}(\mathfrak{g}) / \mathcal{D}(\mathfrak{g}) \cdot \text{ad } \mathfrak{g}$ as an $(\mathcal{D}(\mathfrak{g}), \mathcal{D}(\mathfrak{t})^W)$ -bimodule. It is easy to see that, taking global sections on each side of formula (1.2.2), yields an isomorphism, cf. [LS3, p. 1109],

$$\Gamma(\mathfrak{g} \times \mathfrak{t}, \mathcal{M}) \stackrel{\Xi}{=} [\mathcal{D}(\mathfrak{g}) / \mathcal{D}(\mathfrak{g}) \cdot \text{ad } \mathfrak{g}] \otimes_{\mathcal{D}(\mathfrak{t})^W} \mathcal{D}(\mathfrak{t}). \quad (2.4.1)$$

Here, the object on the left has the structure of a *left* $\mathcal{D}(\mathfrak{g}) \otimes \mathcal{D}(\mathfrak{t})$ -module and the object on the right has the structure of a $(\mathcal{D}(\mathfrak{g}), \mathcal{D}(\mathfrak{t}))$ -bimodule. These structures are related via the isomorphism Ξ as follows $\Xi[(u \otimes v)m] = u\Xi(m)v^\top$, for any $m \in \Gamma(\mathfrak{g} \times \mathfrak{t}, \mathcal{M})$ and any $u \otimes v \in \mathcal{D}(\mathfrak{g}) \otimes \mathcal{D}(\mathfrak{t})$ where $v \mapsto v^\top$ is an *anti*-involution of the algebra $\mathcal{D}(\mathfrak{t})$ given by $t \mapsto t, \frac{\partial}{\partial t} \mapsto -\frac{\partial}{\partial t}$.

According to formula (1.2.2) the Harish-Chandra module has the form $\mathcal{M} = \mathcal{D} / \mathcal{I}$ where \mathcal{I} is a left ideal of \mathcal{D} . The order filtration on \mathcal{D} restricts to a filtration on \mathcal{I} and it also induces a quotient filtration $F_\bullet^{\text{ord}} \mathcal{M}$ on $\mathcal{D} / \mathcal{I}$. Using the identifications $T^*(\mathfrak{g} \times \mathfrak{t}) = \mathfrak{G} \times \mathfrak{T}$ and $\widetilde{\text{gr}}^{\text{ord}} \mathcal{D} = \mathcal{O}_{\mathfrak{G} \times \mathfrak{T}}$, we get $\widetilde{\text{gr}}^{\text{ord}} \mathcal{M} = \mathcal{O}_{\mathfrak{G} \times \mathfrak{T}} / \widetilde{\text{gr}}^{\text{ord}} \mathcal{I}$ where $\widetilde{\text{gr}}^{\text{ord}} \mathcal{I}$, the associated graded ideal, is a subsheaf of ideals of $\mathcal{O}_{\mathfrak{G} \times \mathfrak{T}}$, not necessarily reduced, in general. A relation between $\widetilde{\text{gr}}^{\text{ord}} \mathcal{I}$ and $\mathcal{J} \subset \mathcal{O}_{\mathfrak{G} \times \mathfrak{T}}$, the ideal sheaf of the (non reduced) subscheme $\mathfrak{C} \times_{\mathfrak{C} // G} \mathfrak{T} \subset \mathfrak{G} \times \mathfrak{T}$, is provided by part (i) of the lemma below.

Notation 2.4.2. We put $\delta_{\mathfrak{t}} := \prod_{\alpha \in R^+} \alpha$ and $t^r := \mathfrak{t} \cap \mathfrak{g}^r$. Let $dx \in \mathcal{K}_{\mathfrak{g}}$, resp. $dt \in \mathcal{K}_{\mathfrak{t}}$, be a constant volume form on \mathfrak{g} , resp. on \mathfrak{t} . Thus, $dx dt$ is a section of $\mathcal{K}_{\mathfrak{g} \times \mathfrak{t}}$.

Write $\mathbf{j} : \mathfrak{r} \hookrightarrow \mathfrak{g} \times \mathfrak{t}$ for the locally closed imbedding and let $\mathbf{j}_! \mathcal{O}_{\mathfrak{r}}$ be the minimal extension, see [Bo], of the structure sheaf $\mathcal{O}_{\mathfrak{r}}$ viewed as a $\mathcal{D}_{\mathfrak{r}}$ -module.

Lemma 2.4.3. (i) *One has inclusions $\mathcal{J} \subset \widetilde{\text{gr}}^{\text{ord}} \mathcal{I} \subset \sqrt{\mathcal{J}}$.*

(ii) *The Harish-Chandra module \mathcal{M} is a simple holonomic \mathcal{D} -module, specifically, one has an isomorphism $\mathcal{M} \cong \mathbf{j}_! \mathcal{O}_{\mathfrak{r}}$ and an equality $[\text{Supp}(\widetilde{\text{gr}}^{\text{ord}} \mathcal{M})] = [\mathfrak{X}]$ of algebraic cycles in $\mathfrak{G} \times \mathfrak{T}$.*

Proof. To simplify notation, it will be convenient below to work with spaces of global sections rather than with sheaves. Thus, we put $I = \Gamma(\mathfrak{g} \times \mathfrak{t}, \mathcal{I}) \subset \mathcal{D}(\mathfrak{g} \times \mathfrak{t})$, resp. $J = \Gamma(\mathfrak{G} \times \mathfrak{T}, \mathcal{J}) \subset \mathbb{C}[\mathfrak{G} \times \mathfrak{T}]$. The vector space $\mathfrak{g} \times \mathfrak{t}$ being affine, it is sufficient to prove an analogue of the lemma for the ideals $\text{gr } I$ and J where throughout the proof we put $\text{gr} := \text{gr}^{\text{ord}}$.

Equip the space $\mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g}) \cdot \text{ad } \mathfrak{g}$ with the quotient filtration. Then, taking associated graded spaces on each side of isomorphism (2.4.1) yields the following chain of graded maps, where pr stands for the natural projection

$$\left(\frac{\text{gr } \mathcal{D}(\mathfrak{g})}{\text{gr}[\mathcal{D}(\mathfrak{g}) \cdot \text{ad } \mathfrak{g}]} \right) \otimes_{\text{gr } \mathcal{D}(\mathfrak{t})^W} \text{gr } \mathcal{D}(\mathfrak{t}) \xrightarrow{pr} \text{gr} \left[\left(\frac{\mathcal{D}(\mathfrak{g})}{\mathcal{D}(\mathfrak{g}) \cdot \text{ad } \mathfrak{g}} \right) \otimes_{\mathcal{D}(\mathfrak{t})^W} \mathcal{D}(\mathfrak{t}) \right] \xrightarrow{\text{gr } \Xi} \text{gr } \Gamma(\mathfrak{g} \times \mathfrak{t}, \mathcal{M}).$$

We have $\text{gr } \mathcal{D}(\mathfrak{g}) = \mathbb{C}[\mathfrak{G}]$. Observe that the map $\text{gr}(\text{ad}) : \mathfrak{g} \rightarrow \text{gr } \mathcal{D}(\mathfrak{g})$ may be identified with the map $\kappa^* : \mathfrak{g} = \mathfrak{g}^* \rightarrow \mathbb{C}[\mathfrak{G}]$, the pull-back morphism induced by the commutator map κ . By definition, one has $\mathbb{C}[\mathfrak{G}]/\mathbb{C}[\mathfrak{G}] \cdot \kappa^*(\mathfrak{g}) = \mathbb{C}[\mathfrak{C}]$. Therefore, using the inclusion $\text{gr } \mathcal{D}(\mathfrak{g}) \cdot \text{gr}(\text{ad})(\mathfrak{g}) \subset \text{gr}[\mathcal{D}(\mathfrak{g}) \cdot \text{ad } \mathfrak{g}]$ we obtain a chain of graded maps

$$\mathbb{C}[\mathfrak{C}] = \mathbb{C}[\mathfrak{G}]/\mathbb{C}[\mathfrak{G}] \cdot \kappa^*(\mathfrak{g}) = \text{gr } \mathcal{D}(\mathfrak{g})/\text{gr } \mathcal{D}(\mathfrak{g}) \cdot \text{gr}(\text{ad})(\mathfrak{g}) \twoheadrightarrow \text{gr } \mathcal{D}(\mathfrak{g})/\text{gr}[\mathcal{D}(\mathfrak{g}) \cdot \text{ad } \mathfrak{g}]. \quad (2.4.4)$$

We let a denote the composite map in (2.4.4).

Now, the radial part map $\text{rad} : \mathcal{D}(\mathfrak{g})^G \rightarrow \mathcal{D}(\mathfrak{t})^W$ is known to respect the order filtrations, [Wa, §3]. Moreover, $\text{gr}(\text{rad}) : \text{gr } \mathcal{D}(\mathfrak{g})^G \rightarrow \text{gr } \mathcal{D}(\mathfrak{t})^W$, the associated graded map, is nothing but the algebra map $\text{res} : \mathbb{C}[\mathfrak{G}]^G \rightarrow \mathbb{C}[\mathfrak{T}]^W$ in (1.3.1). Thus, combining the maps in the two displayed formulas above and writing $b : \mathbb{C}[\mathfrak{T}] \xrightarrow{\sim} \text{gr } \mathcal{D}(\mathfrak{t})$ for the standard isomorphism, we obtain the following graded surjective maps

$$\mathbb{C}[\mathfrak{C}] \otimes_{\mathbb{C}[\mathfrak{T}]^W} \mathbb{C}[\mathfrak{T}] \xrightarrow{a \otimes b} \left(\frac{\text{gr } \mathcal{D}(\mathfrak{g})}{\text{gr}[\mathcal{D}(\mathfrak{g}) \cdot \text{ad } \mathfrak{g}]} \right) \otimes_{\text{gr } \mathcal{D}(\mathfrak{t})^W} \text{gr } \mathcal{D}(\mathfrak{t}) \xrightarrow{\text{gr } \Xi \circ pr} \text{gr } \Gamma(\mathfrak{g} \times \mathfrak{t}, \mathcal{M}).$$

Further, by definition, we have $\mathbb{C}[\mathfrak{G} \times \mathfrak{T}]/J = \mathbb{C}[\mathfrak{C} \times_{\mathfrak{C} // G} \mathfrak{T}] = \mathbb{C}[\mathfrak{C}] \otimes_{\mathbb{C}[\mathfrak{C} // G]} \mathbb{C}[\mathfrak{T}]$. Thus, the composite map in the last formula gives a graded surjective morphism

$$\mathbb{C}[\mathfrak{G} \times \mathfrak{T}]/J = \mathbb{C}[\mathfrak{C}] \otimes_{\mathbb{C}[\mathfrak{C} // G]} \mathbb{C}[\mathfrak{T}] \xrightarrow{\text{gr } \Xi \circ pr \circ (a \otimes b)} \text{gr } \Gamma(\mathfrak{g} \times \mathfrak{t}, \mathcal{M}) = \text{gr } \mathcal{D}(\mathfrak{g} \times \mathfrak{t})/\text{gr } I.$$

This proves the inclusion $J \subset \text{gr } I$. Hence, set theoretically, we have $\text{Supp}(\tilde{\text{gr}}^{\text{ord}} \mathcal{M}) \subset \mathfrak{X}$.

We know that \mathfrak{X}^{rs} is an open dense *smooth* subset of $\mathfrak{C} \times_{\mathfrak{C} // G} \mathfrak{T}$, by Lemma 2.1.3(i). Hence, the ideal J is generically reduced. Further, we know that \mathfrak{X} is a Lagrangian subvariety (Lemma 2.1.5) and that the dimension of any irreducible component of the support of the sheaf $\tilde{\text{gr}} \mathcal{M}$ is $\geq \dim \mathfrak{X}$. We conclude that either $\mathcal{M} = 0$ or else we have that $[\text{Supp}(\tilde{\text{gr}} \mathcal{M})] = [\mathfrak{X}]$, so \mathfrak{X} is the only irreducible component of $\text{Supp}(\tilde{\text{gr}} \mathcal{M})$ and this component occurs with multiplicity 1. In the latter case, \mathcal{M} must be a simple holonomic \mathcal{D} -module, by Lemma 2.2.1. Below, we mimic the proof of [HK1, Proposition 4.7.1] to show that $\mathcal{M} \neq 0$.

Let $pr_{\mathfrak{g}}$ and $pr_{\mathfrak{t}}$ denote the natural projections of \mathfrak{r}^{rs} to \mathfrak{g}^{rs} and to \mathfrak{t}^r , respectively. The restriction of the imbedding \mathbf{j} to the open dense subset $\mathfrak{r}^{rs} \subset \mathfrak{r}$ gives a *closed* imbedding $\mathbf{j}^{rs} : \mathfrak{r}^{rs} \hookrightarrow \mathfrak{g}^{rs} \times \mathfrak{t}^r$. According to parts (i) and (iii) of Proposition 3.1.1 (of §3.1 below), there is a nowhere vanishing G -invariant section $\omega \in \mathcal{K}_{\mathfrak{r}^{rs}}$ of the canonical bundle. The element $(dx \, dt)^{-1} \otimes (\mathbf{j}^{rs})_* \omega$ is therefore a nonzero G -invariant section of the \mathcal{D} -module $\int_{\mathfrak{r}^{rs}} \mathcal{O}_{\mathfrak{r}^{rs}}$ on $\mathfrak{g}^{rs} \times \mathfrak{t}^r$, see §2.2 and [HTT, p.22].

The radial part map rad is known to have the following property, [HC]:

$$\delta_{\mathfrak{t}} \cdot u(f)|_{\mathfrak{t}} = (\text{rad } u)(\delta_{\mathfrak{t}} \cdot f|_{\mathfrak{t}}), \quad \forall u \in \mathcal{D}(\mathfrak{g})^G, \, f \in \mathbb{C}[\mathfrak{g}]^G. \quad (2.4.5)$$

Using the above formula and an equation $pr_{\mathfrak{g}}^*(dx) = (pr_{\mathfrak{t}}^*\delta_{\mathfrak{t}}) \cdot \omega$, see Proposition 3.1.1(iii) of §3 below, one verifies easily that the section $(dx dt)^{-1} \otimes (\mathfrak{j}^{rs})_* \omega$ is annihilated by the ideal $\mathcal{I}|_{\mathfrak{g}^{rs} \times \mathfrak{t}^r}$. Therefore, similarly to [HK1, §4.7], we conclude that the assignment $\mathcal{D}_{\mathfrak{g}^{rs} \times \mathfrak{t}^r} \rightarrow \int_{\mathfrak{j}^{rs}} \mathcal{O}_{\mathfrak{t}^{rs}}, 1 \mapsto (dx dt)^{-1} \otimes (\mathfrak{j}^{rs})_* \omega$, descends to a well defined nonzero \mathcal{D} -module morphism

$$\mathcal{M}|_{\mathfrak{g}^{rs} \times \mathfrak{t}^r} = (\mathcal{D}/\mathcal{I})|_{\mathfrak{g}^{rs} \times \mathfrak{t}^r} \longrightarrow \int_{\mathfrak{j}^{rs}} \mathcal{O}_{\mathfrak{t}^{rs}}. \quad (2.4.6)$$

We conclude that $\mathcal{M} \neq 0$. It follows, as we have shown above, that \mathcal{M} is a simple \mathcal{D} -module. Hence, the map (2.4.6) must be an isomorphism. Moreover, we have $\mathcal{M} \cong \mathfrak{j}_{!*}^{rs} \mathcal{O}_{\mathfrak{t}^{rs}} = \mathfrak{j}_{!*} \mathcal{O}_{\mathfrak{t}}$. Part (ii) of Lemma 2.4.3 follows.

To complete the proof of part (i), we observe that the section ω provides a trivialization of the canonical bundle $\mathcal{K}_{\mathfrak{t}^{rs}}$. This implies, thanks to the isomorphism in (2.4.6) and formula (2.2.2), that we have $(\widetilde{\text{gr}} \mathcal{M})|_{\mathfrak{X}^{rs}} \cong \mathcal{O}_{N_{\mathfrak{t}^{rs}}}$. Hence, any function $f \in \text{gr } I$ viewed as a function on $\mathfrak{G} \times \mathfrak{T}$ vanishes on the set \mathfrak{X}^{rs} . The set \mathfrak{X}^{rs} being Zariski dense in $\mathfrak{C} \times_{\mathfrak{C}/G} \mathfrak{T}$, by Lemma 2.1.3(i), we deduce that the function f vanishes on the zero set of the ideal J . Hence, $f \in \sqrt{J}$ by Hilbert's Nullstellensatz. The inclusion $\text{gr } I \subset \sqrt{J}$ follows. \square

2.5. Hodge filtration on the Harish-Chandra module. The minimal extension $\mathfrak{j}_{!*} \mathcal{O}_{\mathfrak{t}}$ has a canonical structure of Hodge \mathcal{D} -module, [Sa], p.857, Corollary 2. This makes the Harish-Chandra module \mathcal{M} a Hodge module via the isomorphism of Lemma 2.4.3(ii). Let $F^{\text{Hodge}} \mathcal{M}$ be the Hodge filtration on \mathcal{M} and $\widetilde{\text{gr}}^{\text{Hodge}} \mathcal{M}$ be an associated graded sheaf. Observe further that the \mathcal{D} -module $\mathfrak{j}_{!*} \mathcal{O}_{\mathfrak{t}}$ is isomorphic to its Verdier dual so, we have $\mathbb{D}(\mathcal{M}) \cong \mathcal{M}$.

Thus, from the discussion of §2.3 and Lemma 2.4.3(ii), we conclude

Corollary 2.5.1. *The sheaf $\widetilde{\text{gr}}^{\text{Hodge}} \mathcal{M}$ is a Cohen-Macaulay coherent $\mathcal{O}_{\mathfrak{G} \times \mathfrak{T}}$ -module which is isomorphic to its Grothendieck-Serre dual, up to a shift. In addition, we have $\text{Supp}(\widetilde{\text{gr}}^{\text{Hodge}} \mathcal{M}) = \mathfrak{X}$.*

Remark 2.5.2. The normalization of the Hodge filtration that we use in this paper differs by a degree shift from the one used by Saito [Sa].

Our normalization is determined by the requirement that, for any closed immersion $f : X \hookrightarrow Y$, of smooth varieties, the Hodge filtration on the right \mathcal{D}_Y -module $\int_f^R \mathcal{K}_X$ be equal to the order filtration introduced at the end of section 2.2. Thus, we have $F_{-1}^{\text{Hodge}}(\int_f^R \mathcal{K}_X) = 0$, which is not the case in Saito's normalization, see [Sa, formula (3.2.2.3) and Lemma 5.1.9] and [HTT, p. 222].

Degree shifts clearly do not affect the validity of Corollary 2.5.1. \diamond

In the previous subsection, we have considered the order filtration on the Harish-Chandra module \mathcal{M} . The isomorphism in (2.4.6) implies, in view of Remark 2.5.2, that the order and the Hodge filtrations agree on the open dense subset $\mathfrak{g}^{rs} \times \mathfrak{t}^r$. We do not know if these two filtrations agree on the whole of $\mathfrak{g} \times \mathfrak{t}$.

The following result, to be proved in §4.6, provides a partial answer.

Proposition 2.5.3. *With our normalization of the Hodge filtration on \mathcal{M} , for any $k \geq 0$, one has an inclusion $F_k^{\text{ord}} \mathcal{M} \subset F_k^{\text{Hodge}} \mathcal{M}$.*

According to Lemma 2.4.3, the isomorphism of Theorem 1.3.3 fits into the following chain of morphisms of coherent sheaves on $\mathfrak{G} \times \mathfrak{T}$:

$$\mathcal{O}_{\mathfrak{C} \times_{\mathfrak{C}/G} \mathfrak{T}} \longrightarrow \widetilde{\text{gr}}^{\text{ord}} \mathcal{M} \longrightarrow \mathcal{O}_{\mathfrak{X}} \hookrightarrow \psi_* \mathcal{O}_{\mathfrak{X}_{\text{norm}}} \xrightarrow{\text{Theorem 1.3.3}} \widetilde{\text{gr}}^{\text{Hodge}} \mathcal{M}.$$

2.6. Outline of proof of Theorems 1.3.3 and 1.3.4. We begin with the following standard result.

Lemma 2.6.1. *Let X be an irreducible scheme and $\psi : X_{\text{norm}} \rightarrow X_{\text{red}}$ the normalization map. Let $j : U \hookrightarrow X$ be a Zariski open imbedding such that U is smooth and the set $X \setminus U$ has codimension ≥ 2 in X . Let \mathcal{F} be a Cohen-Macaulay sheaf on X such that $j^*\mathcal{F} \cong \mathcal{O}_U$.*

Then, there are natural isomorphisms of \mathcal{O}_X -modules

$$\mathcal{F} \cong j_*\mathcal{O}_U \cong \psi_*\mathcal{O}_{X_{\text{norm}}}. \quad (2.6.2)$$

Proof. Let $Z = X \setminus U$ and let $\mathcal{H}_Z^k(\mathcal{F})$ denote the k th cohomology sheaf of \mathcal{F} with support in Z . One has a standard long exact sequence of cohomology with support

$$0 \longrightarrow \mathcal{H}_Z^0(\mathcal{F}) \longrightarrow \mathcal{F} \xrightarrow{a} j_*j^*\mathcal{F} \longrightarrow \mathcal{H}_Z^1(\mathcal{F}) \longrightarrow \dots$$

where j_* stands for a (nonderived) sheaf theoretic push-forward and a is the canonical adjunction morphism.

A maximal Cohen-Macaulay sheaf has no nonzero torsion subsheaves, [E, §21.4]. Therefore, the sheaf \mathcal{F} is actually an $\mathcal{O}_{X_{\text{red}}}$ -module. In addition, we have $\mathcal{H}_Z^0(\mathcal{F}) = 0$. Further, a well known general result says that, for any maximal Cohen-Macaulay sheaf \mathcal{F} on X and any closed subscheme $Z \subset X$, one has a vanishing $\mathcal{H}_Z^k(\mathcal{F}) = 0$ for all $k < \dim X - \dim Z$ (the result can be reduced to the case of a local ring where it follows e.g. from [E, Theorem A.4.3]). Applying this result in our case and using the codimension ≥ 2 assumption of the lemma, we get that $\mathcal{H}_Z^1(\mathcal{F}) = 0$. Thus, the morphism a in the long exact sequence above is an isomorphism. We deduce that $\mathcal{F} \cong j_*j^*\mathcal{F} \cong j_*\mathcal{O}_U$, the first isomorphism in (2.6.2).

Observe next that the algebra structure on \mathcal{O}_U makes $j_*\mathcal{O}_U$ an $\mathcal{O}_{X_{\text{red}}}$ -algebra. Let $Y := \text{Spec}(j_*\mathcal{O}_U)$ be the relative spectrum of that algebra. Thus, Y is a scheme equipped with a morphism $f : Y \rightarrow X_{\text{red}}$ that restricts to an isomorphism $f : f^{-1}(U) \xrightarrow{\sim} U$; by definition, we have $j_*\mathcal{O}_U = f_*\mathcal{O}_Y$. The scheme Y is reduced and irreducible, since the algebra $j_*\mathcal{O}_U$ clearly has no zero divisors. The codimension ≥ 2 assumption implies that $j_*\mathcal{O}_U$ is a coherent $\mathcal{O}_{X_{\text{red}}}$ -module hence f is a finite birational morphism. Hence, the set $Y \setminus f^{-1}(U)$ has codimension ≥ 2 in Y . We conclude that the scheme Y is smooth in codimension 1 and that it is Cohen-Macaulay, thanks to the isomorphism $\mathcal{F} \cong j_*\mathcal{O}_U$. It follows, by Serre's criterion, that Y is a normal variety; moreover, $f = \psi$ is the normalization map so, we have $j_*\mathcal{O}_U = f_*\mathcal{O}_Y = \psi_*\mathcal{O}_{X_{\text{norm}}}$. \square

We need the following definition, motivated in part by [Ha1, Lemma 3.6.2].

Definition 2.6.3. Let \mathfrak{C}_i , $i = 1, 2$, be the set of pairs $(x_1, x_2) \in \mathfrak{C}$ such that x_i is a regular element of \mathfrak{g} . Put $\mathfrak{C}^{rr} = \mathfrak{C}_1 \cup \mathfrak{C}_2$.

Clearly, each of the sets \mathfrak{C}_i , $i = 1, 2$, is an open subset of \mathfrak{C}^r . Thus, \mathfrak{C}^{rr} is an open subset of \mathfrak{C} which is contained in the smooth locus of \mathfrak{C} . Furthermore, in §6.3 we will prove

Lemma 2.6.4. *The set $\mathfrak{C} \setminus \mathfrak{C}^{rr}$ has codimension ≥ 2 in \mathfrak{C} .*

Corollary 2.6.5. *The set $\mathfrak{X}^{rr} := p^{-1}(\mathfrak{C}^{rr})$ is a smooth Zariski open subset of \mathfrak{X} ; furthermore, the set $\mathfrak{X} \setminus \mathfrak{X}^{rr}$ has codimension ≥ 2 in \mathfrak{X} .*

Proof of Corollary. Let $\mathfrak{X}_i := p^{-1}(\mathfrak{C}_i)$, $i = 1, 2$. According to Lemma 2.1.5, we have $\mathfrak{X}_1 = N_{\mathfrak{t}}$, a Zariski open subset contained in the smooth locus of \mathfrak{X} . By symmetry, the set \mathfrak{X}_2 is contained in the smooth locus of \mathfrak{X} as well. Therefore, $\mathfrak{X}^{rr} = \mathfrak{X}_1 \cup \mathfrak{X}_2$ is a smooth Zariski open subset of \mathfrak{X} . Furthermore, the map p being finite, it follows from Lemma 2.6.4 that the set $\mathfrak{X} \setminus \mathfrak{X}^{rr}$ has codimension ≥ 2 in \mathfrak{X} . \square

Write $j : \mathfrak{X}^{rr} \hookrightarrow \mathfrak{X}$ for the open imbedding. A key role in the proof of Theorem 1.3.3 is played by the following result

Proposition 2.6.6. *There is a natural isomorphism $j^*(\tilde{\mathrm{gr}}^{\mathrm{Hodge}} \mathcal{M}) \cong \mathcal{O}_{\mathfrak{X}^{rr}}$.*

The proof of this proposition will occupy most of sections 3 and 4. Our approach is based on the Hotta-Kashiwara construction of \mathcal{M} via the Springer resolution, see §4.5.

Remark 2.6.7. We observe that the isomorphism $\mathcal{M} = j_{!*}\mathcal{O}_{\mathfrak{t}}$ of Lemma 2.4.3 and the fact that the canonical bundle $\mathcal{K}_{\mathfrak{t}}$ is trivial (see Proposition 3.1.1(i), (iii)) imply an isomorphism $(\tilde{\mathrm{gr}}^{\mathrm{Hodge}} \mathcal{M})|_{N_{\mathfrak{t}}} \cong \mathcal{O}_{N_{\mathfrak{t}}}$ of coherent sheaves, see (2.2.2). Note further that we have $\mathfrak{X}_2 = \sigma(\mathfrak{X}_1)$ where σ is the following involution

$$\sigma : \mathfrak{G} \times \mathfrak{T} \leftrightarrow \mathfrak{G} \times \mathfrak{T}, \quad (x_1, x_2) \times (t_1, t_2) \leftrightarrow (x_2, x_1) \times (t_2, t_1). \quad (2.6.8)$$

Therefore, it would be tempting to try to deduce Proposition 2.6.6 from the isomorphism $(\tilde{\mathrm{gr}}^{\mathrm{Hodge}} \mathcal{M})|_{N_{\mathfrak{t}}} \cong \mathcal{O}_{N_{\mathfrak{t}}}$ by proving an isomorphism $\sigma^*(\tilde{\mathrm{gr}}^{\mathrm{Hodge}} \mathcal{M}) \cong \tilde{\mathrm{gr}}^{\mathrm{Hodge}} \mathcal{M}$. Such an approach is motivated by the observation that the \mathcal{D} -module \mathcal{M} is isomorphic to its Fourier transform in the sense of \mathcal{D} -modules, which is an immediate consequence of Definition 1.2.1. The functor $\sigma^* : \mathrm{Coh} T^*(\mathfrak{g} \times \mathfrak{t}) \rightarrow \mathrm{Coh} T^*(\mathfrak{g} \times \mathfrak{t})$ may be viewed as a ‘classical analogue’ of the Fourier transform of \mathcal{D} -modules on $\mathfrak{g} \times \mathfrak{t}$. Thus, Proposition 2.6.6 would follow from the invariance of \mathcal{M} under the Fourier transform, had we known a general result saying that the functor $\tilde{\mathrm{gr}}^{\mathrm{Hodge}}(-)$ on $(\mathbb{C}^\times\text{-monodromic})$ Hodge \mathcal{D} -modules on a vector space, commutes with Fourier transform. Unfortunately, such a result is not available at the moment of writing of this paper. Indeed, this seems to be a very difficult question (we are grateful to C. Sabbah for information on this subject). \diamond

Proposition 2.6.6 and Lemma 2.6.4 imply Theorem 1.3.3. Let $\mathcal{J}^{\mathrm{Hodge}} \subset \mathcal{O}_{\mathfrak{G} \times \mathfrak{T}}$ be the annihilator of $\tilde{\mathrm{gr}}^{\mathrm{Hodge}} \mathcal{M}$ viewed as an $\mathcal{O}_{\mathfrak{G} \times \mathfrak{T}}$ -module and let $X \subset \mathfrak{G} \times \mathfrak{T}$ be a closed subscheme defined by the ideal $\mathcal{J}^{\mathrm{Hodge}}$. By Corollary 2.5.1, the sheaf $\tilde{\mathrm{gr}}^{\mathrm{Hodge}} \mathcal{M}$ is Cohen-Macaulay and, set theoretically, one has $X = \mathfrak{X}$. Further, the isomorphism $\mathcal{M} \cong j_{!*}\mathcal{O}_{\mathfrak{t}}$ of Lemma 2.4.3 implies that the ideal $\mathcal{J}^{\mathrm{Hodge}}$ is generically reduced. Hence, X is reduced and we have $X = \mathfrak{X}$, since \mathfrak{X} is reduced by definition.

Thus, thanks to Corollary 2.6.5, we are in the setting of Lemma 2.6.1, where we take $\mathcal{F} = \tilde{\mathrm{gr}}^{\mathrm{Hodge}} \mathcal{M}$ and let $j : U = \mathfrak{X}^{rr} \hookrightarrow X = \mathfrak{X}$ be the open imbedding. We see that Theorem 1.3.3 is a direct consequence of Lemma 2.6.1 combined with Proposition 2.6.6. \square

Theorem 1.3.3 implies Theorem 1.3.4. By Theorem 1.3.3, we have $\tilde{\mathrm{gr}}^{\mathrm{Hodge}} \mathcal{M} \cong \psi_*\mathcal{O}_{\mathfrak{X}_{\mathrm{norm}}}$. Further, thanks to Corollary 2.5.1, we know that the sheaf $\tilde{\mathrm{gr}}^{\mathrm{Hodge}} \mathcal{M}$ is Cohen-Macaulay and, moreover, it is isomorphic to its Grothendieck-Serre dual, up to a shift. Thus, the sheaf $\psi_*\mathcal{O}_{\mathfrak{X}_{\mathrm{norm}}}$ has similar properties. Since, Grothendieck’s duality commutes with finite morphisms, we deduce that $\mathcal{O}_{\mathfrak{X}_{\mathrm{norm}}}$ is a Cohen-Macaulay sheaf which is, moreover, isomorphic to its Grothendieck-Serre dual, that is, to the dualizing sheaf $\mathcal{K}_{\mathfrak{X}_{\mathrm{norm}}}$, up to a shift. Therefore, we have $\mathcal{O}_{\mathfrak{X}_{\mathrm{norm}}} \cong \mathcal{K}_{\mathfrak{X}_{\mathrm{norm}}}$, completing the proof. \square

3. SPRINGER RESOLUTIONS

3.1. An analogue of the Grothendieck-Springer resolution. Let \mathcal{B} be the flag variety, the variety of all Borel subalgebras $\mathfrak{b} \subset \mathfrak{g}$. Motivated by Grothendieck and Springer, we introduce the following incidence varieties

$$\tilde{\mathfrak{g}} := \{(\mathfrak{b}, x) \in \mathcal{B} \times \mathfrak{g} \mid x \in \mathfrak{b}\}, \quad \text{resp.} \quad \tilde{\mathfrak{g}} := \{(\mathfrak{b}, x, y) \in \mathcal{B} \times \mathfrak{g} \times \mathfrak{g} \mid x, y \in \mathfrak{b}\}.$$

The first projection makes $\tilde{\mathfrak{g}}$, resp. $\tilde{\mathfrak{G}}$, a sub vector bundle of the trivial vector bundle $\mathcal{B} \times \mathfrak{g} \rightarrow \mathcal{B}$, resp. $\mathcal{B} \times \mathfrak{G} \rightarrow \mathcal{B}$. Given a Borel subgroup $B \subset G$ with Lie algebra $\mathfrak{b} = \text{Lie } B$, we have $\mathcal{B} \cong G/B$. That gives a G -equivariant vector bundle isomorphism $\tilde{\mathfrak{g}} \cong G \times_B \mathfrak{b}$, resp. $\tilde{\mathfrak{G}} \cong G \times_B (\mathfrak{b} \times \mathfrak{b})$. Thus, $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{G}}$ are smooth connected varieties.

Recall that, for any pair $\mathfrak{b}, \mathfrak{b}'$ of Borel subalgebras of \mathfrak{g} , there is a canonical isomorphism $\mathfrak{b}/[\mathfrak{b}, \mathfrak{b}] \cong \mathfrak{b}'/[\mathfrak{b}', \mathfrak{b}']$, cf. eg. [CG, Lemma 3.1.26]. Given a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{b}$, the composite $\mathfrak{t} \hookrightarrow \mathfrak{b} \twoheadrightarrow \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]$ yields an isomorphism $\mathfrak{t} \xrightarrow{\sim} \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]$. Therefore, the assignment $(\mathfrak{b}', x) \mapsto x \bmod [\mathfrak{b}', \mathfrak{b}'] \in \mathfrak{b}'/[\mathfrak{b}', \mathfrak{b}']$, resp. $(\mathfrak{b}', x, y) \mapsto (x \bmod [\mathfrak{b}', \mathfrak{b}'], y \bmod [\mathfrak{b}', \mathfrak{b}']) \in \mathfrak{b}'/[\mathfrak{b}', \mathfrak{b}'] \times \mathfrak{b}'/[\mathfrak{b}', \mathfrak{b}']$, gives a well defined smooth morphism $\nu : \tilde{\mathfrak{g}} \rightarrow \mathfrak{t}$, resp. $\nu : \tilde{\mathfrak{G}} \rightarrow \mathfrak{T}$.

Finally, we introduce a projective G -equivariant morphism $\mu : \tilde{\mathfrak{g}} \rightarrow \mathfrak{g}$, $(\mathfrak{b}, x) \mapsto x$, resp. $\mu : \tilde{\mathfrak{G}} \rightarrow \mathfrak{G}$, $(\mathfrak{b}, x, y) \mapsto (x, y)$. The map μ is known as the *Grothendieck-Springer resolution*.

Proposition 3.1.1. (i) *The image of the map $\mu \times \nu : \tilde{\mathfrak{g}} \rightarrow \mathfrak{g} \times \mathfrak{t}$ is contained in $\mathfrak{x} = \mathfrak{g} \times_{\mathfrak{g}/G} \mathfrak{t}$. The resulting morphism $\pi : \tilde{\mathfrak{g}} \rightarrow \mathfrak{x}$ is a resolution of singularities, so $\pi^{-1}(\mathfrak{r}) \xrightarrow{\sim} \mathfrak{r}$ is an isomorphism.*

(ii) *The image of the map $\mu \times \nu : \tilde{\mathfrak{G}} \rightarrow \mathfrak{G} \times \mathfrak{T}$ is contained in $\mathfrak{G} \times_{\mathfrak{G}/G} \mathfrak{T}$. The resulting map $\tilde{\mathfrak{G}} \rightarrow [\mathfrak{G} \times_{\mathfrak{G}/G} \mathfrak{T}]_{\text{red}}$ is a proper birational morphism.*

(iii) *The canonical bundle on $\tilde{\mathfrak{g}}$ has a natural trivialization by a nowhere vanishing G -invariant section $\omega \in \mathcal{K}_{\tilde{\mathfrak{g}}}$ such that one has $\mu^*(dx) = (\nu^*\delta_{\mathfrak{t}}) \cdot \omega$, cf. Notation 2.4.2.*

(iv) *We have $\mathcal{K}_{\tilde{\mathfrak{G}}} = \wedge^{\dim \mathcal{B}} \mathfrak{q}^* \mathcal{T}_{\mathcal{B}}$ (here $\mathcal{T}_{\mathcal{B}}$ is the tangent sheaf and $\mathfrak{q} : \tilde{\mathfrak{G}} \rightarrow \mathcal{B}$ is the projection).*

Part (i) of Proposition 3.1.1 is well-known, cf. [BB], [CG]. The descriptions of canonical bundles in parts (iii)-(iv) are straightforward. The equation of part (iii) that involves the section ω appears eg. in [HK1], formula (4.1.4). This equation is, in essence, nothing but Weyl's classical integration formula.

Part (ii) of Proposition 3.1.1 is an immediate consequence of Proposition 6.1.2, of §6.1 below. Lemma 6.1.1 implies, in particular, that the image of the map μ equals the set of pairs $(x, y) \in \mathfrak{G}$ such that x and y generate a solvable Lie subalgebra of \mathfrak{g} . We remark also that the statement of Proposition 3.1.1(ii) is a variation of results concerning the null-fiber of the adjoint quotient map $\mathfrak{G} \rightarrow \mathfrak{G}/G$, see [Ri2], [KW]. \square

Question 3.1.2. Is the variety $[\mathfrak{G} \times_{\mathfrak{G}/G} \mathfrak{T}]_{\text{red}}$ normal, resp. Cohen-Macaulay?

3.2. Symplectic geometry interpretation. The map $(\mathfrak{b}, x) \mapsto (\mathfrak{b}, x, \nu(\mathfrak{b}, x))$ gives a closed immersion $\epsilon : \tilde{\mathfrak{g}} \hookrightarrow \mathcal{B} \times \mathfrak{g} \times \mathfrak{t}$. The image of this immersion is a smooth subvariety $\epsilon(\tilde{\mathfrak{g}}) \subset \mathcal{B} \times \mathfrak{g} \times \mathfrak{t}$. Let Λ be the total space of the conormal bundle of that subvariety. Thus, Λ is a smooth \mathbb{C}^\times -stable Lagrangian subvariety in $T^*(\mathcal{B} \times \mathfrak{g} \times \mathfrak{t}) = T^*\mathcal{B} \times (\mathfrak{G} \times \mathfrak{T})$. Let $\text{pr}_{\Lambda \rightarrow T^*\mathcal{B}} : \Lambda \rightarrow T^*\mathcal{B}$, resp. $\text{pr}_{\Lambda \rightarrow \mathfrak{G} \times \mathfrak{T}} : \Lambda \rightarrow \mathfrak{G} \times \mathfrak{T}$, denote the restriction to Λ of the projection of $T^*\mathcal{B} \times \mathfrak{G} \times \mathfrak{T}$ to the first, resp. along the first, factor.

Let \mathcal{N} be the variety of nilpotent elements of \mathfrak{g} and put $\tilde{\mathcal{N}} := \{(\mathfrak{b}, x) \in \mathcal{B} \times \mathfrak{g} \mid x \in [\mathfrak{b}, \mathfrak{b}]\}$. There is a natural isomorphism $T^*\mathcal{B} \cong \tilde{\mathcal{N}}$ of G -equivariant vector bundles on \mathcal{B} that identifies the cotangent space at a point $\mathfrak{b} \in \mathcal{B}$ with the vector space $(\mathfrak{g}/\mathfrak{b})^* \cong [\mathfrak{b}, \mathfrak{b}]$, cf. [CG, ch. 3]. Let $\Phi : \tilde{\mathcal{N}} \xrightarrow{\sim} T^*\mathcal{B}$ be an isomorphism obtained by composing the above isomorphism with the sign involution along the fibers of the vector bundle $T^*\mathcal{B}$.

Restricting the commutator map κ to each Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ yields a morphism $\tilde{\kappa} : \tilde{\mathfrak{G}} \rightarrow \tilde{\mathcal{N}}$, $(\mathfrak{b}, x, y) \mapsto (\mathfrak{b}, [x, y])$. Let Ψ denote the map $(\Phi \circ \tilde{\kappa}) \times \mu \times \nu : \tilde{\mathfrak{G}} \rightarrow T^*\mathcal{B} \times \mathfrak{G} \times \mathfrak{T}$.

Proposition 3.2.1. *The map Ψ yields an isomorphism $\tilde{\mathfrak{G}} \xrightarrow{\sim} \Lambda$ that fits into a commutative diagram*

$$\begin{array}{ccccc} \mathfrak{G} \times \mathfrak{T} & \xleftarrow{\mu \times \nu} & \tilde{\mathfrak{G}} & \xrightarrow{\tilde{\kappa}} & \tilde{\mathcal{N}} \\ \parallel \text{Id} & & \parallel \Psi & & \parallel \Phi \\ \mathfrak{G} \times \mathfrak{T} & \xleftarrow{\text{pr}_{\Lambda \rightarrow \mathfrak{G} \times \mathfrak{T}}} & \Lambda & \xrightarrow{\text{pr}_{\Lambda \rightarrow T^* \mathcal{B}}} & T^* \mathcal{B} \end{array} \quad (3.2.2)$$

Proof. Fix $\mathfrak{b} \in \mathcal{B}$ and $x \in \mathfrak{b}$. So, $(\mathfrak{b}, x) \in \tilde{\mathfrak{g}}$ and the corresponding point in $\mathcal{B} \times \mathfrak{g} \times \mathfrak{t}$ is given by the triple $u = (\mathfrak{b}, x, x \bmod [\mathfrak{b}, \mathfrak{b}])$. The fiber of the tangent bundle $T(\mathcal{B} \times \mathfrak{g} \times \mathfrak{t})$ at the point u may be identified with the vector space $T_u(\mathcal{B} \times \mathfrak{g} \times \mathfrak{t}) = (\mathfrak{g}/\mathfrak{b}) \times \mathfrak{g} \times \mathfrak{t}$. Hence, the tangent space to the submanifold $\epsilon(\tilde{\mathfrak{g}})$ equals

$$T_u(\epsilon(\tilde{\mathfrak{g}})) = \{(\alpha \bmod \mathfrak{b}, [\alpha, x] + \beta, \beta \bmod [\mathfrak{b}, \mathfrak{b}]) \in (\mathfrak{g}/\mathfrak{b}) \times \mathfrak{g} \times \mathfrak{t} \mid \alpha \in \mathfrak{g}, \beta \in \mathfrak{b}\}.$$

Now, write $\langle -, - \rangle$ for an invariant bilinear form on \mathfrak{g} and use it to identify the fiber of the cotangent bundle $T^*(\mathcal{B} \times \mathfrak{g} \times \mathfrak{t})$ at u with the vector space $[\mathfrak{b}, \mathfrak{b}] \times \mathfrak{g} \times \mathfrak{t}$. Let $(n, y, h) \in [\mathfrak{b}, \mathfrak{b}] \times \mathfrak{g} \times \mathfrak{t}$ be a point of that vector space. Such a point belongs to Λ_u , the fiber at u of the conormal bundle on the subvariety $\epsilon(\tilde{\mathfrak{g}})$, if and only if the following equation holds

$$\langle \alpha, n \rangle + \langle [\alpha, x] + \beta, y \rangle + \langle \beta \bmod [\mathfrak{b}, \mathfrak{b}], h \rangle = 0 \quad \forall \alpha \in \mathfrak{g}, \beta \in \mathfrak{b}. \quad (3.2.3)$$

Taking $\alpha = 0$ and applying equation (3.2.3) we get $\langle [\mathfrak{b}, \mathfrak{b}], y \rangle = 0$. Hence, $y \in \mathfrak{b}$ and $h = -y \bmod [\mathfrak{b}, \mathfrak{b}]$. Next, for any $\alpha \in \mathfrak{g}$, we have $\langle [\alpha, x], y \rangle = \langle \alpha, [x, y] \rangle$. Hence, for $\beta = 0$ and any $n \in [\mathfrak{b}, \mathfrak{b}]$, $y \in \mathfrak{b}$, equation (3.2.3) gives

$$0 = \langle \alpha, n \rangle + \langle [\alpha, x], y \rangle = \langle \alpha, n + [x, y] \rangle \quad \forall \alpha \in \mathfrak{g}.$$

It follows that $n + [x, y] = 0$. We conclude that

$$\Lambda_u = \{(-[x, y], y, y \bmod [\mathfrak{b}, \mathfrak{b}]) \in [\mathfrak{b}, \mathfrak{b}] \times \mathfrak{g} \times \mathfrak{t}, \quad y \in \mathfrak{b}\}. \quad (3.2.4)$$

We have a projection $\varpi : \tilde{\mathfrak{G}} \rightarrow \tilde{\mathfrak{g}}, (\mathfrak{b}, x, y) \rightarrow (\mathfrak{b}, x)$ along the last factor. The vector space in right hand side of (3.2.4) is equal to the image of the set $\varpi^{-1}(\mathfrak{b}, x)$ under the map $\Psi = (\Phi \circ \tilde{\kappa}) \times \mu \times \nu$. We conclude that the map Ψ gives an isomorphism of vector bundles $\tilde{\mathfrak{G}} \rightarrow \tilde{\mathfrak{g}}$ and $\Lambda \rightarrow \epsilon(\tilde{\mathfrak{g}})$, respectively. \square

3.3. The scheme $\tilde{\mathfrak{X}}$. We use the notation $\mathfrak{z} : X \hookrightarrow T^*X$ for the zero section of the cotangent bundle on a variety X .

We consider the following commutative diagram where the vertical map $\mu|_{\tilde{\mathcal{N}}}$ is known as the Springer resolution,

$$\begin{array}{ccccccc} \mathcal{B} = \mu^{-1}(0) & \xrightarrow{\mathfrak{z}: \mathfrak{b} \mapsto (\mathfrak{b}, 0)} & \tilde{\mathcal{N}} & \xleftarrow{\tilde{\kappa}} & \tilde{\mathfrak{G}} & \xrightarrow{\nu} & \mathfrak{T} \\ \downarrow \mu & & \downarrow \mu|_{\tilde{\mathcal{N}}} & & \downarrow \mu & & \downarrow \\ \{0\} & \hookrightarrow & \mathcal{N} & \hookrightarrow & \mathfrak{g} & \xleftarrow{\kappa} & \mathfrak{G} \longrightarrow \mathfrak{G} \times_{\mathfrak{G}/G} \mathfrak{T} \end{array} \quad (3.3.1)$$

Let $\tilde{\mathfrak{X}} := \tilde{\kappa}^{-1}(\mathfrak{z}(\mathcal{B})) \subset \tilde{\mathfrak{G}}$, a scheme theoretic preimage of the zero section. Set theoretically, one has

$$\tilde{\mathfrak{X}} = \{(\mathfrak{b}, x, y) \in \mathcal{B} \times \mathfrak{g} \times \mathfrak{g} \mid x, y \in \mathfrak{b}, [x, y] = 0\}. \quad (3.3.2)$$

Diagram (3.3.1) shows that the morphism μ maps $\tilde{\mathfrak{X}}$ to \mathfrak{C} , resp. $\mu \times \nu$ maps $\tilde{\mathfrak{X}}$ to $\mathfrak{C} \times_{\mathfrak{C}/G} \mathfrak{T}$. It follows from Proposition 3.2.1 that the map Ψ induces an isomorphism of schemes

$$\tilde{\mathfrak{X}} = \tilde{\kappa}^{-1}(\mathfrak{z}(\mathcal{B})) \xrightarrow{\Psi} \text{pr}_{\Lambda \rightarrow T^* \mathcal{B}}^{-1}(\mathfrak{z}(\mathcal{B})) = \Lambda \cap (\mathcal{B} \times \mathfrak{G} \times \mathfrak{T})$$

where the scheme structure on each side is that of a scheme theoretic preimage.

Let $\tilde{\mathfrak{X}}^{rr} := \mu^{-1}(\mathfrak{C}^{rr})$, a Zariski open subset of the scheme $\tilde{\mathfrak{X}} = \tilde{\kappa}^{-1}(\mathfrak{z}(\mathcal{B}))$.

Lemma 3.3.3. *The differential of the morphism $\tilde{\kappa} : \tilde{\mathfrak{G}} \rightarrow \tilde{\mathcal{N}}$ is surjective at any point $(\mathfrak{b}, x, y) \in \tilde{\mathfrak{X}}^{rr}$; in particular, the set $\tilde{\mathfrak{X}}^{rr}$ is contained in the smooth locus of the scheme $\tilde{\mathfrak{X}}$.*

Proof. Let $x \in \mathfrak{b}$ and write $\text{ad}_{\mathfrak{b}} x$ for the map $\mathfrak{b} \mapsto [\mathfrak{b}, \mathfrak{b}]$, $u \mapsto [x, u]$. One has $\dim \mathfrak{b} - \dim \mathfrak{g}_x \leq \dim \mathfrak{b} - \dim \text{Ker}(\text{ad}_{\mathfrak{b}} x) = \dim \text{Im}(\text{ad}_{\mathfrak{b}} x) \leq \dim[\mathfrak{b}, \mathfrak{b}]$. For $x \in \mathfrak{b} \cap \mathfrak{g}^r$, we have $\dim \mathfrak{g}_x = \dim \mathfrak{t}$ and the above inequalities yield $\dim[\mathfrak{b}, \mathfrak{b}] = \dim \mathfrak{b} - \dim \mathfrak{g}_x \leq \dim(\text{Im ad}_{\mathfrak{b}} x) \leq \dim[\mathfrak{b}, \mathfrak{b}]$. Thus, in this case, $\text{Im}(\text{ad}_{\mathfrak{b}} x) = [\mathfrak{b}, \mathfrak{b}]$ i.e. the map $\text{ad}_{\mathfrak{b}} x$ is surjective.

Now, let $(\mathfrak{b}, x, y) \in \tilde{\mathfrak{X}}^{rr}$. Without loss of generality, we may assume that x is a regular element of \mathfrak{g} . The differential of the commutator map $\kappa : \mathfrak{b} \times \mathfrak{b} \rightarrow [\mathfrak{b}, \mathfrak{b}]$ at the point $(x, y) \in \mathfrak{b} \times \mathfrak{b}$ is a linear map $d_{\mathfrak{b}, x, y} \kappa : \mathfrak{b} \oplus \mathfrak{b} \rightarrow [\mathfrak{b}, \mathfrak{b}]$ given by the formula $d_{\mathfrak{b}, x, y} \kappa : (u, v) \mapsto \text{ad}_{\mathfrak{b}} x(u) - \text{ad}_{\mathfrak{b}} y(v)$. We see that $\text{Im}(\text{ad}_{\mathfrak{b}} x) \subseteq \text{Im}(d_{\mathfrak{b}, x, y} \kappa)$. By the preceding paragraph, we deduce that the map $d_{\mathfrak{b}, x, y} \kappa$ is surjective. The lemma follows from this by G -equivariance. \square

Remark 3.3.4. The scheme $\tilde{\mathfrak{X}}$ is not irreducible, in general, cf. [Ba]. So, the open set $\tilde{\mathfrak{X}}^{rr}$ is not necessarily dense in $\tilde{\mathfrak{X}}$. \diamond

We use Proposition 3.2.1 to identify the set $\tilde{\mathfrak{X}}^{rr} \subset \tilde{\mathfrak{X}} \subset \tilde{\mathfrak{G}}$ with a subset of $\Lambda \cap (\mathcal{B} \times \mathfrak{G} \times \mathfrak{T})$. Recall the notation $\mathfrak{X}^{rr} = p^{-1}(\mathfrak{C}^{rr})$.

Corollary 3.3.5. (i) *The varieties Λ and $\mathcal{B} \times \mathfrak{G} \times \mathfrak{T}$ meet transversely at any point of $\tilde{\mathfrak{X}}^{rr}$, so $\tilde{\mathfrak{X}}^{rr}$ is a smooth Zariski open subset of $\Lambda \cap (\mathcal{B} \times \mathfrak{G} \times \mathfrak{T})$.*

(ii) *The morphism $\pi : \tilde{\mathfrak{X}}^{rr} \rightarrow \mathfrak{X}^{rr}$, the restriction of the morphism $\mu \times \nu$ to the set $\tilde{\mathfrak{X}}^{rr}$, is an isomorphism of algebraic varieties.*

Proof. Part (i) is equivalent to the statement of Lemma 3.3.3. To prove (ii), let X denote the preimage of \mathfrak{C}^{rr} under the first projection $\mathfrak{C} \times_{\mathfrak{C}/G} \mathfrak{T} \rightarrow \mathfrak{C}$, so we have $X_{\text{red}} = \mathfrak{X}^{rr}$. It is clear that the morphism $\mu \times \nu$ maps $\tilde{\mathfrak{X}}^{rr}$ to X . We claim that the resulting map $\pi : \tilde{\mathfrak{X}}^{rr} \rightarrow X$ is a set theoretic bijection. Indeed, it is surjective, by Lemma 2.1.3(i), since the image of this map contains the set $\mathfrak{C}^{rs} \times_{\mathfrak{C}/G} \mathfrak{T}$ and $\mu \times \nu$, hence also π , is a proper morphism. To prove injectivity, we interpret the map $(\mathfrak{b}, x, y) \mapsto (x, y)$ as a composition of the imbedding $\tilde{\mathfrak{X}} \hookrightarrow \tilde{\mathfrak{g}} \times \mathfrak{g}$, $(\mathfrak{b}, x, y) \mapsto (\mathfrak{b}, x) \times y$ and the map $\pi \times \text{Id}_{\mathfrak{g}} : \tilde{\mathfrak{g}} \times \mathfrak{g} \rightarrow \mathfrak{r} \times \mathfrak{g}$. This last map gives a bijection between the set $\mathfrak{r} \times \mathfrak{g}$ and its preimage in $\tilde{\mathfrak{g}} \times \mathfrak{g}$, thanks to Proposition 3.1.1(i). Our claim follows.

Recall that $\tilde{\mathfrak{X}}^{rr}$ is a smooth scheme by Lemma 3.3.3. Hence, the scheme theoretic image of $\tilde{\mathfrak{X}}^{rr}$ under the morphism π is actually contained in $\mathfrak{X}^{rr} = X_{\text{red}}$. The reduced scheme \mathfrak{X}^{rr} is smooth by Corollary 2.6.5. Thus, $\pi : \tilde{\mathfrak{X}}^{rr} \rightarrow \mathfrak{X}^{rr}$ is a morphism of smooth varieties, which is a set theoretic bijection. Such a morphism is necessarily an isomorphism, by Zariski's main theorem, and part (ii) follows.

There is an alternative proof that the morphism $\pi : \tilde{\mathfrak{X}}^{rr} \rightarrow \mathfrak{X}^{rr}$ is étale based on symplectic geometry. In more detail, put $X = \mathcal{B}$ and $Y = \mathfrak{g} \times \mathfrak{t}$ and let $\epsilon : \tilde{\mathfrak{g}} \hookrightarrow X \times Y$ be the imbedding, cf. §3.2. We have smooth locally closed subvarieties $\mathfrak{r} \subset \mathfrak{g} \times \mathfrak{t}$, $\pi^{-1}(\mathfrak{r}) \subset \tilde{\mathfrak{g}}$ and $Z := \epsilon(\pi^{-1}(\mathfrak{r})) \subset X \times Y$, respectively. We use Proposition 2.1.5, resp. Proposition 3.2.1, to identify \mathfrak{X}_1 with $N_{\mathfrak{r}}$, resp. $\pi^{-1}(\mathfrak{g}^r \times \mathfrak{g}) \cap \Lambda$ with N_Z .

We know that the projection $X \times Y \rightarrow Y$ induces an isomorphism $Z \xrightarrow{\sim} \mathfrak{r}$, by Proposition 3.1.1(i). Now, one can prove a general result saying that, in this case, the map $(X \times T^*Y) \cap N_Z \rightarrow N_{\mathfrak{r}}$ induced by the projection $T^*X \times T^*Y \rightarrow T^*Y$ is étale at any point where N_Z meets the subvariety $X \times T^*Y \subset T^*(X \times Y)$ transversely. The latter condition holds in our

case thanks to part (i) of the Corollary. This implies the isomorphism $\pi : \pi^{-1}(\mathfrak{C}_1) \xrightarrow{\sim} \mathfrak{X}_1$. The isomorphism $\pi : \pi^{-1}(\mathfrak{C}_2) \xrightarrow{\sim} \mathfrak{X}_2$ then follows by symmetry. \square

3.4. A DG algebra. In this subsection, we construct a sheaf \mathcal{A} of DG $\mathcal{O}_{\tilde{\mathfrak{G}}}$ -algebras such that $\mathcal{H}^0(\mathcal{A})$, the zero cohomology sheaf, is isomorphic to the structure sheaf of the closed subscheme $\tilde{\mathfrak{X}} \subset \tilde{\mathfrak{G}}$. To explain the construction, let $\mathcal{T} := \mathcal{T}_{\mathcal{B}}$ and write $\iota : \mathcal{B} \hookrightarrow T^*\mathcal{B}$ for the zero section, resp. $q : T^*\mathcal{B} \rightarrow \mathcal{B}$ for the projection.

The sheaf $q^*\mathcal{T}^*$ on $T^*\mathcal{B}$ comes equipped with a canonical Euler section eu such that, for each covector $\xi \in T^*\mathcal{B}$, the value of eu at the point ξ is equal to ξ . Further, there is a standard Koszul complex $\dots \rightarrow \wedge^3 q^*\mathcal{T} \rightarrow \wedge^2 q^*\mathcal{T} \rightarrow q^*\mathcal{T} \rightarrow 0$ with differential ∂_{eu} given by contraction with eu . The complex $(\wedge^\bullet q^*\mathcal{T}, \partial_{\text{eu}})$ is a locally free resolution of the sheaf $\iota_*\mathcal{O}_{\mathcal{B}}$ on $T^*\mathcal{B}$.

We may use the isomorphism $T^*\mathcal{B} \cong \tilde{\mathcal{N}}$ to view the morphism $\tilde{\kappa}$ as a map $\tilde{\mathfrak{G}} \rightarrow T^*\mathcal{B}$. Hence, the pull-back of the Koszul complex above via the map $\tilde{\kappa}$ is a complex of locally free sheaves on $\tilde{\mathfrak{G}}$ that represents the object $L\tilde{\kappa}^*(\iota_*\mathcal{O}_{\mathcal{B}}) \in D_{\text{coh}}^b(\tilde{\mathfrak{G}})$.

Let $q : \tilde{\mathfrak{G}} \rightarrow \mathcal{B}$ denote the first projection, so we have $\tilde{\kappa}^*q^*\mathcal{T} = q^*\mathcal{T}$ and one may identify $\tilde{\kappa}^*\text{eu}$, the pull-back of the section eu , with a section of $q^*\mathcal{T}^*$. For each $n \geq 0$, let $\mathcal{A}_n = \wedge^n q^*\mathcal{T} = \wedge^n \tilde{\kappa}^*q^*\mathcal{T}$, where $\mathcal{A}_0 := \mathcal{O}_{\tilde{\mathfrak{G}}}$. Contraction with $\tilde{\kappa}^*\text{eu}$ gives a differential $\partial_{\tilde{\kappa}^*\text{eu}} : \mathcal{A}_\bullet \rightarrow \mathcal{A}_{\bullet-1}$. Thus, we may (and will) view $\mathcal{A} := \bigoplus_{n \geq 0} \mathcal{A}_n$ as a sheaf of coherent DG $\mathcal{O}_{\tilde{\mathfrak{G}}}$ -algebras, with multiplication given by the wedge product and with the differential $\partial_{\tilde{\kappa}^*\text{eu}}$, a graded derivation of degree (-1) .

Notation 3.4.1. Write $\mathcal{H}^j(\mathcal{F}) \in \text{Coh } X$ for the j th cohomology sheaf of an object $\mathcal{F} \in D_{\text{coh}}^b(X)$.

By construction, one has $\mathcal{H}^0(\mathcal{A}, \partial_{\tilde{\kappa}^*\text{eu}}) = \mathcal{O}_{\tilde{\mathfrak{X}}}$, cf. (3.3.2). Thus, one may view \mathcal{A} as the structure sheaf of a certain DG scheme, a “derived analogue” of the scheme $\tilde{\mathfrak{X}}$.

The DG algebra \mathcal{A} has also appeared, in an implicit form, in a calculation in [SV].

Remark 3.4.2. The DG algebra \mathcal{A} is concentrated in degrees $0 \leq i \leq d := \dim \mathcal{B}$ and we have $\mathcal{A}_d = \mathcal{K}_{\tilde{\mathfrak{G}}}$, by Proposition 3.1.1(iv). It follows that \mathcal{A} is a *self-dual* DG algebra in the sense that multiplication in \mathcal{A} yields an isomorphism of complexes

$$\mathcal{A}_{d-\bullet} \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_{\tilde{\mathfrak{G}}}}(\mathcal{A}_\bullet, \mathcal{A}_d) = \mathcal{H}om_{\mathcal{O}_{\tilde{\mathfrak{G}}}}(\mathcal{A}_\bullet, \mathcal{K}_{\tilde{\mathfrak{G}}}).$$

4. PROOF OF THE MAIN THEOREM

4.1. The Hotta-Kashiwara construction. The structure sheaf $\mathcal{O}_{\tilde{\mathfrak{g}}}$ has an obvious structure of holonomic left $\mathcal{D}_{\tilde{\mathfrak{g}}}$ -module. So, one has $\int_{\mu \times \nu} \mathcal{O}_{\tilde{\mathfrak{g}}}$, the direct image of this $\mathcal{D}_{\tilde{\mathfrak{g}}}$ -module via the projective morphism $\mu \times \nu$, cf. §3.1. Each cohomology group $\mathcal{H}^k(\int_{\mu \times \nu} \mathcal{O}_{\tilde{\mathfrak{g}}})$ is a holonomic \mathcal{D} -module set theoretically supported on the variety $\mathfrak{r} \subset \mathfrak{g} \times \mathfrak{t}$.

Hotta and Kashiwara proved the following important result, [HK1, Theorem 4.2].

Theorem 4.1.1. *For any $k \neq 0$, we have $\mathcal{H}^k(\int_{\mu \times \nu} \mathcal{O}_{\tilde{\mathfrak{g}}}) = 0$; furthermore, there is a natural isomorphism $\mathcal{H}^0(\int_{\mu \times \nu} \mathcal{O}_{\tilde{\mathfrak{g}}}) \cong \mathbf{j}_! \mathcal{O}_{\mathfrak{r}}$ of \mathcal{D} -modules.*

The \mathcal{D} -module $\mathcal{H}^0(\int_{\mu \times \nu} \mathcal{O}_{\tilde{\mathfrak{g}}})$ has a canonical nonzero section $(dx dt)^{-1} \otimes (\mu \times \nu)_*\omega$, cf. Proposition 3.1.1(iii). Hotta and Kashiwara [HK1, §4.7] show that this section is annihilated by the left ideal

$$\mathcal{I}' := \mathcal{D} \cdot (\text{ad } \mathfrak{g} \otimes 1) + \mathcal{D} \cdot \{P - \text{rad}(P) \mid P \in \mathbb{C}[\mathfrak{g}]^G\} + \mathcal{D} \cdot \{Q - \text{rad}(Q) \mid Q \in (\text{Sym } \mathfrak{g})^G\}$$

(we have exploited their argument in the proof of Lemma 2.4.3).

Furthermore, it is proved in [HK1, §4.7] that \mathcal{D}/\mathcal{I}' is a simple \mathcal{D} -module and the assignment $u \mapsto u[(dx\,dt)^{-1} \otimes (\mu \times \nu)_*\omega]$ yields a \mathcal{D} -module isomorphism $\mathcal{D}/\mathcal{I}' \xrightarrow{\sim} \mathcal{H}^0(\int_{\mu \times \nu} \mathcal{O}_{\tilde{\mathfrak{g}}})$. An alternative, more direct, proof of an analogous isomorphism may be found in [HK2].

Remark 4.1.2. Hotta and Kashiwara defined the Harish-Chandra module as the quotient \mathcal{D}/\mathcal{I}' . Comparing formula (1.2.2) with the definition of the ideal \mathcal{I}' we see that one has $\mathcal{I}' \subseteq \mathcal{I}$. This inclusion of left ideals yields a surjection $\mathcal{D}/\mathcal{I}' \twoheadrightarrow \mathcal{D}/\mathcal{I} = \mathcal{M}$. Since \mathcal{D}/\mathcal{I}' is a simple \mathcal{D} -module, this surjection must be an isomorphism. So, Definition 1.2.1 is equivalent, *a posteriori*, to the one used by Hotta and Kashiwara.

4.2. Direct image for filtered \mathcal{D} -modules. Let $f : X \rightarrow Y$ a proper morphism. The direct image functor \int_f can be upgraded to a functor $D_{coh}^b(\mathcal{F}\mathcal{D}_X) \rightarrow D_{coh}^b(\mathcal{F}\mathcal{D}_Y)$, $(M, F) \mapsto \int_f(M, F)$ between filtered derived categories, cf. [La, §4], [Sa, §2.3]. The latter functor is known to commute with the associated graded functor $\widetilde{\text{dgr}}(-)$. We will only need a special case of this result for maps of the form $f : X \times Y \rightarrow Y$, the projection along a proper variety X . In such a case, one has a diagram

$$T^*Y \xleftarrow{\varpi} X \times T^*Y \xrightarrow{\varsigma = \mathfrak{z} \times \text{Id}_{T^*Y}} T^*X \times T^*Y = T^*(X \times Y). \quad (4.2.1)$$

Here, ϖ denotes the second projection and \mathfrak{z} is the zero section.

The relation between the functors $\widetilde{\text{dgr}}(-)$ and \int_f is provided by the following result, see [La], formula (5.6.1.2).

Theorem 4.2.2. *Let $f : X \times Y \rightarrow Y$ be the second projection where X is proper. Then, for any $(M, F) \in D_{coh}^b(\mathcal{F}\mathcal{D}_{X \times Y})$, in $D_{coh}^b(T^*Y)$ there is a functorial isomorphism*

$$\widetilde{\text{dgr}}\left(\int_f(M, F)\right) = R\varpi_*[(\mathcal{K}_X \boxtimes \mathcal{O}_{T^*Y}) \otimes_{\mathcal{O}_{X \times T^*Y}} L\varsigma^* \widetilde{\text{dgr}}(M, F)]. \quad \square$$

The cohomology sheaves $\mathcal{H}^*(\int_f(M, F))$ of the filtered complex $\int_f(M, F)$ come equipped with an induced filtration. However, Theorem 4.2.2 is not sufficient, in general, for describing $\widetilde{\text{gr}}\mathcal{H}^*(\int_f(M, F))$, the associated graded sheaves. A theorem of Saito stated below says that, in the case of Hodge modules, Theorem 4.2.2 is indeed sufficient for that.

Let $f : X \rightarrow Y$ be a proper morphism of smooth varieties. A Hodge module on X may be viewed as an object $(M, F) \in D_{coh}^b(\mathcal{F}\mathcal{D}_X)$, so there is a well defined object $\int_f(M, F) \in D_{coh}^b(\mathcal{F}\mathcal{D}_Y)$.

One of the main results of Saito's theory reads, see [Sa, Theorem 5.3.1]:

Theorem 4.2.3. *For any $(M, F) \in \text{HM}(X)$ and any projective morphism $f : X \rightarrow Y$, the filtered complex $\int_f(M, F)$ is strict, cf. Definition 2.3.1, and each cohomology group $\mathcal{H}^j(\int_f(M, F))$ has the natural structure of a Hodge module on Y .*

In the situation of the theorem, we refer to the induced filtration on $\mathcal{H}^j(\int_f(M, F))$, $j = 0, 1, \dots$, as the Hodge filtration and let $\widetilde{\text{gr}}^{\text{Hodge}}\mathcal{H}^j(\int_f(M, F))$ denote the associated graded coherent sheaf on T^*Y . Similar notation will be used for right \mathcal{D} -modules.

4.3. Key result. We recall the setup of §3.2. Thus, we have the immersion $\epsilon : \widetilde{\mathfrak{g}} \hookrightarrow \mathcal{B} \times \mathfrak{g} \times \mathfrak{t}$, $(\mathfrak{b}, x) \mapsto (\mathfrak{b}, x, \nu(\mathfrak{b}, x))$ and we write $\Lambda \subset T^*(\mathcal{B} \times \mathfrak{g} \times \mathfrak{t})$ for the total space of the conormal bundle on $\epsilon(\widetilde{\mathfrak{g}})$. We will view the structure sheaf \mathcal{O}_Λ as a coherent sheaf on $T^*\mathcal{B} \times \mathfrak{G} \times \mathfrak{T}$ supported on Λ .

In the special case where $X = \mathcal{B}$ and $Y = \mathfrak{g} \times \mathfrak{t}$ diagram (4.2.1) takes the form

$$\Lambda \subset T^*\mathcal{B} \times \mathfrak{G} \times \mathfrak{T} \xleftarrow{\varsigma} \mathcal{B} \times \mathfrak{G} \times \mathfrak{T} \xrightarrow{\varpi} \mathfrak{G} \times \mathfrak{T}. \quad (4.3.1)$$

Theorem 4.3.2. *All nonzero cohomology sheaves of the object $R\varpi_* L\varsigma^* \mathcal{O}_\Lambda \in D_{\text{coh}}^b(\mathfrak{G} \times \mathfrak{T})$ vanish and, we have*

$$\mathcal{H}^0(R\varpi_* L\varsigma^* \mathcal{O}_\Lambda) = \widetilde{\text{gr}}^{\text{Hodge}} \mathcal{M}.$$

Proof. Let $E := \int_{\epsilon}^R \mathcal{K}_{\widetilde{\mathfrak{g}}}$, a right Hodge \mathcal{D} -module on $\mathcal{B} \times \mathfrak{g} \times \mathfrak{t}$. Using the notation of Remark 2.5.2, from (2.2.2), we obtain $\text{gr}^{\text{Hodge}} E = q^*(\epsilon_* \mathcal{K}_{\widetilde{\mathfrak{g}}}) = q^*(\epsilon_* \mathcal{O}_{\widetilde{\mathfrak{g}}})$, where in the last equality we have used that the canonical bundle on $\widetilde{\mathfrak{g}}$ is trivial, see Proposition 3.1.1(iii). Therefore, for $\int_{\epsilon} \mathcal{O}_{\widetilde{\mathfrak{g}}} = \mathcal{K}_{\mathcal{B} \times \mathfrak{g} \times \mathfrak{t}}^{-1} \otimes E$, the corresponding left $\mathcal{D}_{\mathcal{B} \times \mathfrak{g} \times \mathfrak{t}}$ -module, we find

$$L\varsigma^*(\widetilde{\text{gr}}^{\text{Hodge}}(\int_{\epsilon} \mathcal{O}_{\widetilde{\mathfrak{g}}})) = L\varsigma^* q^*(\mathcal{K}_{\mathcal{B} \times \mathfrak{g} \times \mathfrak{t}}^{-1} \otimes \epsilon_* \mathcal{O}_{\widetilde{\mathfrak{g}}}) = L\varsigma^* q^*(\mathcal{K}_{\mathcal{B}}^{-1} \boxtimes \mathcal{O}_{\mathfrak{g} \times \mathfrak{t}}) = \mathcal{K}_{\mathcal{B}}^{-1} \otimes L\varsigma^* \mathcal{O}_\Lambda, \quad (4.3.3)$$

where we have used simplified notation $\mathcal{K}_{\mathcal{B}}^{-1} \otimes (-) = (\mathcal{K}_{\mathcal{B}}^{-1} \boxtimes \mathcal{O}_{\mathfrak{G} \times \mathfrak{T}}) \otimes_{\mathcal{O}_{\mathcal{B} \times \mathfrak{G} \times \mathfrak{T}}} (-)$.

We factor the map $\mu \times \nu$ as a composition of the closed imbedding ϵ and a proper projection $f : \mathcal{B} \times \mathfrak{g} \times \mathfrak{t} \rightarrow \mathfrak{g} \times \mathfrak{t}$ along the first factor. We get $\int_{\mu \times \nu} \mathcal{O}_{\widetilde{\mathfrak{g}}} = \int_f (\int_{\epsilon} \mathcal{O}_{\widetilde{\mathfrak{g}}})$. Hence, applying Theorem 4.2.2 to the \mathcal{D} -module $M = \int_{\epsilon} \mathcal{O}_{\widetilde{\mathfrak{g}}}$ and using (4.3.3), we obtain $\widetilde{\text{dgr}}(\int_{\mu \times \nu} \mathcal{O}_{\widetilde{\mathfrak{g}}}) = \widetilde{\text{dgr}}(\int_f (\int_{\epsilon} \mathcal{O}_{\widetilde{\mathfrak{g}}})) = R\varpi_* L\varsigma^* \mathcal{O}_\Lambda$. Thus, by Theorem 4.2.3 applied to the Hodge module $\mathcal{O}_{\widetilde{\mathfrak{g}}}$, we get $\widetilde{\text{gr}}^{\text{Hodge}} \mathcal{H}^j(\int_{\mu \times \nu} \mathcal{O}_{\widetilde{\mathfrak{g}}}) = \mathcal{H}^j(\widetilde{\text{dgr}} \int_{\mu \times \nu} \mathcal{O}_{\widetilde{\mathfrak{g}}}) = \mathcal{H}^j(R\varpi_* L\varsigma^* \mathcal{O}_\Lambda)$ for any $j \in \mathbb{Z}$. We conclude that $\mathcal{H}^j(R\varpi_* L\varsigma^* \mathcal{O}_\Lambda) = 0$ for any $j \neq 0$, thanks to Theorem 4.1.1.

Finally, we observe that the Hodge structure on the minimal extension \mathcal{D} -module $\mathbf{j}_{!*} \mathcal{O}_{\mathfrak{t}}$ is determined by the Hodge structure on $\mathcal{O}_{\mathfrak{t}}$. The morphism $\mu \times \nu : \widetilde{\mathfrak{g}} \rightarrow \mathfrak{t}$ is generically an isomorphism. It follows that the isomorphism of Theorem 4.1.1 respects the Hodge structures. Therefore, the isomorphism $\mathcal{M} \cong \mathcal{H}^0(\int_{\mu \times \nu} \mathcal{O}_{\widetilde{\mathfrak{g}}})$, which is based on the isomorphism $\mathcal{M} \cong \mathbf{j}_{!*} \mathcal{O}_{\mathfrak{t}}$ of Lemma 2.4.3(ii), also respects the Hodge filtrations. We deduce that $\widetilde{\text{gr}}^{\text{Hodge}} \mathcal{M} = \mathcal{H}^0(R\varpi_* L\varsigma^* \mathcal{O}_\Lambda)$. \square

4.4. Let X be a smooth variety and let $i_Y : Y \hookrightarrow X$, resp. $i_Z : Z \hookrightarrow X$, be closed imbeddings of smooth subvarieties. Below, we will use the following simple

Lemma 4.4.1. *In $D_{\text{coh}}^b(X)$, there are canonical quasi-isomorphisms*

$$(i_Y)_* Li_Y^*[(i_Z)_* \mathcal{O}_Z] \cong (i_Y)_* \mathcal{O}_Y \overset{L}{\otimes}_{\mathcal{O}_X} (i_Z)_* \mathcal{O}_Z \cong (i_Z)_* Li_Z^*[(i_Y)_* \mathcal{O}_Y]. \quad (4.4.2)$$

Proof. Let $\Delta_X : X \rightarrow X \times X$ be the diagonal imbedding and define a map $\Delta_{YX} : Y \rightarrow Y \times X$, $y \mapsto (y, i_Y(y))$. We have a cartesian diagram of closed imbeddings

$$\begin{array}{ccc} Y & \xrightarrow{\Delta_{YX}} & Y \times X \\ i_Y \downarrow & & \downarrow i_Y \times \text{Id}_X \\ X & \xrightarrow{\Delta_X} & X \times X \end{array}$$

For any $\mathcal{F} \in D_{\text{coh}}^b(X)$, we have $[(i_Y)_* \mathcal{O}_Y] \boxtimes \mathcal{F} = (i_Y \times \text{Id}_X)_* f^* \mathcal{F}$, where $f : Y \times X \rightarrow X$ is the second projection. Therefore, we obtain

$$\begin{aligned} [(i_Y)_* \mathcal{O}_Y] \overset{L}{\otimes}_{\mathcal{O}_X} \mathcal{F} &= \Delta_X^*([(i_Y)_* \mathcal{O}_Y] \boxtimes \mathcal{F}) = \Delta_X^*(i_Y \times \text{Id}_X)_*(f^* \mathcal{F}) \\ &= (i_Y)_*[\Delta_{YX}^*(f^* \mathcal{F})] = (i_Y)_*(f \circ \Delta_{YX})^* \mathcal{F} = (i_Y)_* i_Y^* \mathcal{F}, \end{aligned}$$

where the third isomorphism is a consequence of proper base change with respect to the cartesian square above. Taking here $\mathcal{F} := (i_Z)_* \mathcal{O}_Z$ yields the isomorphism on the left of (4.4.2). The isomorphism on the right of (4.4.2) is proved similarly. \square

Associated with diagram (3.3.1), there are derived functors

$$D_{\text{coh}}^b(\mathcal{B}) \xrightarrow{\iota_*} D_{\text{coh}}^b(\tilde{\mathcal{N}}) \xrightarrow{L\tilde{\kappa}^*} D_{\text{coh}}^b(\tilde{\mathcal{G}}) \xrightarrow{R(\mu \times \nu)_*} D_{\text{coh}}^b(\mathcal{G} \times \mathcal{T}). \quad (4.4.3)$$

In §3.4, we considered a Koszul complex $(\wedge^\bullet q^* \mathcal{T}, \partial_{\text{eu}})$, a locally free resolution of the sheaf $\iota_* \mathcal{O}_{\mathcal{B}}$ on $T^* \mathcal{B}$. Therefore, the object $L\tilde{\kappa}^*(\iota_* \mathcal{O}_{\mathcal{B}}) \in D_{\text{coh}}^b(\tilde{\mathcal{G}})$ may be represented by the DG $\mathcal{O}_{\tilde{\mathcal{G}}}$ -algebra $(\mathcal{A}, \partial_{\tilde{\kappa}^* \text{eu}}) = \tilde{\kappa}^*(\wedge^\bullet q^* \mathcal{T}, \partial_{\text{eu}})$ introduced in §3.4.

The following result provides a link between the DG algebra \mathcal{A} and Theorem 4.3.2.

Lemma 4.4.4. *In $D_{\text{coh}}^b(\mathcal{G} \times \mathcal{T})$, there is a natural isomorphism*

$$R\varpi_* L\zeta^*((i_\Lambda)_* \mathcal{O}_\Lambda) \simeq R(\mu \times \nu)_* \mathcal{A}.$$

Proof. To simplify notation, we put $Z := \mathcal{B} \times \mathcal{G} \times \mathcal{T}$. Further, write $pr_{T^* \mathcal{B}}$, resp. $pr_{\mathcal{G} \times \mathcal{T}}$, for the projection of the variety $T^* \mathcal{B} \times \mathcal{G} \times \mathcal{T}$ to the first, resp. along the first, factor. Clearly, we have $\varsigma_* \mathcal{O}_Z = pr_{T^* \mathcal{B}}^*(\iota_* \mathcal{O}_{\mathcal{B}})$. Also, using the notation of diagram (3.2.2), we get $pr_{\Lambda \rightarrow T^* \mathcal{B}} = pr_{T^* \mathcal{B}} \circ i_\Lambda$, resp. $pr_{\Lambda \rightarrow \mathcal{G} \times \mathcal{T}} = i_\Lambda \circ pr_{\mathcal{G} \times \mathcal{T}}$. We deduce a chain of isomorphisms

$$Li_\Lambda^*(\varsigma_* \mathcal{O}_Z) = Li_\Lambda^*[pr_{T^* \mathcal{B}}^*(\iota_* \mathcal{O}_{\mathcal{B}})] = L(i_\Lambda \circ pr_{T^* \mathcal{B}})^*(\iota_* \mathcal{O}_{\mathcal{B}}) = L pr_{\Lambda \rightarrow T^* \mathcal{B}}^*(\iota_* \mathcal{O}_{\mathcal{B}}).$$

Next, we use the isomorphism $\Phi : \tilde{\mathcal{N}} \xrightarrow{\sim} T^* \mathcal{B}$, see §3.2, to identify the imbedding $\mathcal{B} \hookrightarrow \tilde{\mathcal{N}}$ with the zero section $\iota : \mathcal{B} \hookrightarrow T^* \mathcal{B}$. Thus, from the above chain of isomorphisms, using commutative diagram (3.2.2), we get

$$R(pr_{\Lambda \rightarrow \mathcal{G} \times \mathcal{T}})_* Li_\Lambda^*(\varsigma_* \mathcal{O}_Z) = R(pr_{\Lambda \rightarrow \mathcal{G} \times \mathcal{T}})_* L pr_{\Lambda \rightarrow T^* \mathcal{B}}^*(\iota_* \mathcal{O}_{\mathcal{B}}) \cong R(\mu \times \nu)_* L\tilde{\kappa}^*(\iota_* \mathcal{O}_{\mathcal{B}}). \quad (4.4.5)$$

Now, we apply Lemma 4.4.1 in the case where $X = T^* \mathcal{B} \times \mathcal{G} \times \mathcal{T}$ and $Y = \Lambda$. So, we have $i_Z = \varsigma$ and the composite isomorphism in (4.4.2) yields $(i_\Lambda)_* Li_\Lambda^*[\varsigma_* \mathcal{O}_Z] = \varsigma_* L\zeta^*[(i_\Lambda)_* \mathcal{O}_\Lambda]$. Note that $\varpi = \varsigma \circ pr_{\mathcal{G} \times \mathcal{T}}$, so $R\varpi_* = (Rpr_{\mathcal{G} \times \mathcal{T}})_* \varsigma_*$. Hence, we obtain

$$\begin{aligned} R\varpi_* L\zeta^*[(i_\Lambda)_* \mathcal{O}_\Lambda] &= R(pr_{\mathcal{G} \times \mathcal{T}})_* \varsigma_* \zeta^*[(i_\Lambda)_* \mathcal{O}_\Lambda] \stackrel{(4.4.2)}{=} R(pr_{\mathcal{G} \times \mathcal{T}})_*(i_\Lambda)_* Li_\Lambda^*[\varsigma_* \mathcal{O}_Z] \\ &\stackrel{(4.4.5)}{=} R(pr_{\Lambda \rightarrow \mathcal{G} \times \mathcal{T}})_* Li_\Lambda^*(\varsigma_* \mathcal{O}_Z) = R(\mu \times \nu)_* L\tilde{\kappa}^*(\iota_* \mathcal{O}_{\mathcal{B}}). \end{aligned}$$

Here, the object on the right is $R(\mu \times \nu)_* \mathcal{A}$, and the lemma is proved. \square

Corollary 4.4.6. *The sheaves $\mathcal{H}^k(R(\mu \times \nu)_* \mathcal{A})$ vanish for all $k \neq 0$ and there is an $\mathcal{O}_{\mathcal{G} \times \mathcal{T}}$ -module isomorphism $\mathcal{H}^0(R(\mu \times \nu)_* \mathcal{A}) \cong \tilde{\text{gr}}^{\text{Hodge}} \mathcal{M}$.*

Proof. This is an immediate consequence of Theorem 4.3.2 and Lemma 4.4.4. \square

4.5. Completing the proof of Theorem 1.3.3. We recall that in order to complete the proof of Theorem 1.3.3 it remains to prove Proposition 2.6.6 and Lemma 2.6.4. The proof of the lemma will be given later, in §6.3.

Proof of Proposition 2.6.6. Let \mathcal{A}_0 be the degree zero homogeneous component of \mathcal{A} . We may view \mathcal{A}_0 as a DG algebra equipped with zero differential and concentrated in degree zero. Thus, one has a natural DG algebra imbedding $f : (\mathcal{A}_0, 0) \hookrightarrow (\mathcal{A}, \partial_{\tilde{\kappa}^* \text{eu}})$.

To simplify notation, put $\pi := \mu \times \nu$. Applying the functor $R\pi_*$ and using that $\mathcal{A}_0 = \mathcal{O}_{\tilde{\mathcal{G}}} = \pi^* \mathcal{O}_{\mathcal{G} \times \mathcal{T}}$, we obtain the following chain of DG algebra morphisms

$$\mathcal{O}_{\mathcal{G} \times \mathcal{T}} \xrightarrow{\text{adjunction}} R\pi_* \pi^* \mathcal{O}_{\mathcal{G} \times \mathcal{T}} = R\pi_* \mathcal{A}_0 \xrightarrow{R\pi_*(f)} R\pi_* \mathcal{A}. \quad (4.5.1)$$

Let $\mathfrak{G}^{rr} \subset \mathfrak{G}$ be a Zariski open subset such that $(\mathfrak{G}^{rr} \times \mathfrak{T}) \cap \mathfrak{X} = \mathfrak{X}^{rr}$. From Lemma 3.3.3 and Corollary 3.3.5(ii), we deduce that the composite morphism in (4.5.1) induces an isomorphism $\mathcal{O}_{\mathfrak{X}^{rr}} \xrightarrow{\sim} R\pi_* \mathcal{A}|_{\mathfrak{G}^{rr} \times \mathfrak{T}}$. Combining the latter isomorphism with the isomorphism $\mathcal{H}^0(R\pi_* \mathcal{A}) \cong \widetilde{\text{gr}}^{\text{Hodge}} \mathcal{M}$ of Corollary 4.4.6 yields the required isomorphism $\mathcal{O}_{\mathfrak{X}^{rr}} \xrightarrow{\sim} j^*(\widetilde{\text{gr}}^{\text{Hodge}} \mathcal{M})$. \square

The statement of the next result was suggested to me by Dmitry Arinkin.

Theorem 4.5.2. *There is $G \times \mathbb{C}^\times \times \mathbb{C}^\times$ -equivariant DG $\mathcal{O}_{\mathfrak{G} \times \mathfrak{T}}$ -algebra quasi-isomorphism: $R(\mu \times \nu)_* \mathcal{A} \simeq \psi_* \mathcal{O}_{\mathfrak{X}_{\text{norm}}}$.*

Proof. The fact that $R(\mu \times \nu)_* \mathcal{A}$ and $\psi_* \mathcal{O}_{\mathfrak{X}_{\text{norm}}}$ are isomorphic as objects of $D_{\text{coh}}^b(\mathfrak{G} \times \mathfrak{T})$ is an immediate consequence of Theorem 1.3.3 and Corollary 4.4.6.

To establish the DG $\mathcal{O}_{\mathfrak{G} \times \mathfrak{T}}$ -algebra quasi-isomorphism, one can argue as follows. Let $\mathcal{F} := \mathcal{H}^0(R(\mu \times \nu)_* \mathcal{A})$. We claim that the sheaf \mathcal{F} is Cohen-Macaulay. This follows from Corollary 4.4.6 since we know that $\widetilde{\text{gr}}^{\text{Hodge}} \mathcal{M}$ is a Cohen-Macaulay coherent sheaf set theoretically supported on \mathfrak{X} , see Corollary 2.5.1.

There is also an alternative proof of the claim that does not use the Cohen-Macaulay property of associated graded sheaves arising from Hodge modules. That alternative proof is based instead on the self duality property of the DG algebra \mathcal{A} , see Remark 3.4.2. The latter property, combined with the fact that the morphism $\mu \times \nu$ is proper, implies that the object $R(\mu \times \nu)_* \mathcal{A} \in D_{\text{coh}}^b(\mathfrak{G} \times \mathfrak{T})$ is isomorphic to its Grothendieck-Serre dual, up to a shift. Therefore, the cohomology vanishing from Corollary 4.4.6 forces the sheaf $\mathcal{F} = \mathcal{H}^0(R(\mu \times \nu)_* \mathcal{A})$ to be Cohen-Macaulay. The claim follows.

Now, according to the proof of Proposition 2.6.6 given above, the composite morphism in (4.5.1) induces a $G \times \mathbb{C}^\times \times \mathbb{C}^\times$ -equivariant $\mathcal{O}_{\mathfrak{G} \times \mathfrak{T}}$ -algebra isomorphism $\mathcal{O}_{\mathfrak{X}^{rr}} \xrightarrow{\sim} j^* \mathcal{F}$. Thus, Lemma 2.6.1 provides a $G \times \mathbb{C}^\times \times \mathbb{C}^\times$ -equivariant algebra isomorphism $\psi_* \mathcal{O}_{\mathfrak{X}_{\text{norm}}} \xrightarrow{\sim} \mathcal{F}$. \square

We observe next that $\mathbb{H}^\bullet(\widetilde{\mathfrak{G}}, \mathcal{A})$, the hyper-cohomology of the DG algebra $(\mathcal{A}, \partial_{\widetilde{\kappa}^* \text{eu}})$, acquires the canonical structure of a graded commutative algebra. From Theorem 4.5.2, for any $k \in \mathbb{Z}$, we deduce

$$\mathbb{H}^k(\widetilde{\mathfrak{G}}, \mathcal{A}) = \mathbb{H}^k(\mathfrak{G} \times \mathfrak{T}, R(\mu \times \nu)_* \mathcal{A}) = H^k(\mathfrak{G} \times \mathfrak{T}, \psi_* \mathcal{O}_{\mathfrak{X}_{\text{norm}}}).$$

The group on the right vanishes for any $k \neq 0$ since the scheme $\mathfrak{G} \times \mathfrak{T}$ is affine. So, we obtain

Corollary 4.5.3. *The hyper-cohomology groups $\mathbb{H}^k(\widetilde{\mathfrak{G}}, \mathcal{A})$ vanish for all $k \neq 0$ and there is a G -equivariant bigraded $\mathbb{C}[\mathfrak{G} \times \mathfrak{T}]$ -algebra isomorphism $\mathbb{H}^0(\widetilde{\mathfrak{G}}, \mathcal{A}) \cong \mathbb{C}[\mathfrak{X}_{\text{norm}}]$.* \square

Remark 4.5.4. Write $\mathcal{X} := \text{Spec } \mathcal{A}$ for the DG scheme associated with the DG algebra \mathcal{A} in the sense of derived algebraic geometry, cf. [TV]. The DG scheme \mathcal{X} may be thought of as a ‘derived analogue’ of the scheme $\widetilde{\mathfrak{X}}$, cf. §3.3. Then, Corollary 4.5.3 says that the morphism $\mu \times \nu$ induces a DG-algebra quasi-isomorphism $\mathbb{C}[\mathfrak{X}_{\text{norm}}] \rightarrow R\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. This may be interpreted as saying that the DG scheme $\text{Spec } \mathcal{A}$ provides, in a sense, a ‘DG resolution’ of the variety $\mathfrak{X}_{\text{norm}}$.

4.6. Proof of Proposition 2.5.3. Let $f : \mathcal{B} \times (\mathfrak{g} \times \mathfrak{t}) \rightarrow \mathfrak{g} \times \mathfrak{t}$ be the second projection.

Lemma 4.6.1. *All nonzero cohomology sheaves of the complex $\int_f^R \mathcal{D}_{\mathcal{B} \times \mathfrak{g} \times \mathfrak{t}}$ vanish and one has an isomorphism $\mathcal{H}^0(\int_f^R \mathcal{D}_{\mathcal{B} \times \mathfrak{g} \times \mathfrak{t}}) \cong \mathcal{D}_{\mathfrak{g} \times \mathfrak{t}}$ of right $\mathcal{D}_{\mathfrak{g} \times \mathfrak{t}}$ -modules.*

Proof. There is a standard Koszul type complex K^\bullet with terms $K^j = \mathcal{D}_{\mathcal{B}} \otimes_{\mathcal{O}_{\mathcal{B}}} \wedge^{-j} \mathcal{T}_{\mathcal{B}}$, $j = 0, -1, \dots, -\dim \mathcal{B}$, that gives a resolution of the structure sheaf $\mathcal{O}_{\mathcal{B}}$, cf. [HTT, Lemma 1.5.27]. Using that $H^0(\mathcal{B}, \mathcal{O}_{\mathcal{B}}) = \mathbb{C}$ and $H^k(\mathcal{B}, \mathcal{O}_{\mathcal{B}}) = 0$ for any $k \neq 0$ we deduce that $R\Gamma(\mathcal{B}, K^\bullet) = R\Gamma(\mathcal{B}, \mathcal{O}_{\mathcal{B}}) = \mathbb{C}$. Therefore, an explicit construction of direct image for *right* \mathcal{D} -modules (see [HTT, Proposition 1.5.28] for its *left* \mathcal{D} -module counterpart) shows that $\int_f^R \mathcal{D}_{\mathcal{B} \times \mathfrak{g} \times \mathfrak{t}} = R\Gamma(\mathcal{B}, K^\bullet) \otimes_{\mathbb{C}} \mathcal{D}_{\mathfrak{g} \times \mathfrak{t}} = \mathcal{D}_{\mathfrak{g} \times \mathfrak{t}}$. \square

Next, we recall the setting of §4.3. We observe that $E = \int_{\epsilon}^R \mathcal{K}_{\tilde{\mathfrak{g}}}$ is a cyclic right $\mathcal{D}_{\mathcal{B} \times \mathfrak{g} \times \mathfrak{t}}$ -module generated by the section $\epsilon_*(\omega \otimes 1)$ where $1 \in \mathcal{D}_{\tilde{\mathfrak{g}} \rightarrow \mathcal{B} \times \mathfrak{g} \times \mathfrak{t}} = \mathcal{D}_{\mathcal{B} \times \mathfrak{g} \times \mathfrak{t}}|_{\tilde{\mathfrak{g}}}$. Therefore, the assignment $1 \mapsto \epsilon_*(\omega \otimes 1)$ can be extended to a surjective morphism $\gamma : \mathcal{D}_{\mathcal{B} \times \mathfrak{g} \times \mathfrak{t}} \twoheadrightarrow E$ of right $\mathcal{D}_{\mathcal{B} \times \mathfrak{g} \times \mathfrak{t}}$ -modules. The quotient filtration on E induced by the projection γ is equal, by our normalization of the Hodge filtration, to the Hodge filtration $F^{\text{Hodge}} E$, see Remark 2.5.2. This filtration on E , resp. the order filtration on $\mathcal{D}_{\mathcal{B} \times \mathfrak{g} \times \mathfrak{t}}$, makes $\int_f^R E$, resp. $\int_f^R \mathcal{D}_{\mathcal{B} \times \mathfrak{g} \times \mathfrak{t}}$, a filtered complex. Applying the functor \int_f^R to γ , a morphism of filtered \mathcal{D} -modules, one obtains a morphism $\int_f^R \gamma : \int_f^R \mathcal{D}_{\mathcal{B} \times \mathfrak{g} \times \mathfrak{t}} \rightarrow \int_f^R E$ of filtered complexes.

We identify the sheaf $\mathcal{D}_{\mathfrak{g} \times \mathfrak{t}}$ with $\mathcal{D}_{\mathfrak{g} \times \mathfrak{t}}^{\text{op}}$ via the trivialization of the canonical bundle on $\mathfrak{g} \times \mathfrak{t}$ provided by the section $dx dt$. Thus, using Lemma 4.6.1, we obtain a chain of morphisms of *left* $\mathcal{D}_{\mathfrak{g} \times \mathfrak{t}}$ -modules

$$\mathcal{D}_{\mathfrak{g} \times \mathfrak{t}} = \mathcal{K}_{\mathfrak{g} \times \mathfrak{t}}^{-1} \otimes \mathcal{H}^0(\int_f^R \mathcal{D}_{\mathcal{B} \times \mathfrak{g} \times \mathfrak{t}}) \xrightarrow{\int_f^R \gamma} \mathcal{K}_{\mathfrak{g} \times \mathfrak{t}}^{-1} \otimes \mathcal{H}^0(\int_f^R E) = \mathcal{H}^0(\int_{\mu \times \nu} \mathcal{O}_{\tilde{\mathfrak{g}}}) = \mathcal{M}. \quad (4.6.2)$$

It is straightforward to see, using the explicit formula $u \mapsto u[(dx dt)^{-1} \otimes (\mu \times \nu)_* \omega]$ for the isomorphism $\mathcal{M} \xrightarrow{\sim} \mathcal{H}^0(\int_{\mu \times \nu} \mathcal{O}_{\tilde{\mathfrak{g}}})$, cf. §4.1, that the composite morphism in (4.6.2) is equal to the natural projection $\mathcal{D}_{\mathfrak{g} \times \mathfrak{t}} \twoheadrightarrow \mathcal{D}_{\mathfrak{g} \times \mathfrak{t}}/\mathcal{I} = \mathcal{M}$.

The proof of Lemma 4.6.1 shows that the filtration on the \mathcal{D} -module $\mathcal{H}^0(\int_f^R \mathcal{D}_{\mathcal{B} \times \mathfrak{g} \times \mathfrak{t}})$ induced by the filtered structure on $\int_f^R \mathcal{D}_{\mathcal{B} \times \mathfrak{g} \times \mathfrak{t}}$ goes, under the isomorphism of the Lemma, to the standard order filtration on the sheaf $\mathcal{D}_{\mathfrak{g} \times \mathfrak{t}}$. It follows that all the maps in (4.6.2) respect the filtrations. Thus, writing $\bar{\gamma} : \mathcal{D}_{\mathfrak{g} \times \mathfrak{t}} \twoheadrightarrow \mathcal{M}$ for the composite map in (4.6.2), we get $\bar{\gamma}(F_k^{\text{ord}} \mathcal{D}_{\mathfrak{g} \times \mathfrak{t}}) \subset F_k^{\text{Hodge}} \mathcal{M}$ for any $k \in \mathbb{Z}$. Proposition 2.6.6 follows from this since the order filtration $F^{\text{ord}} \mathcal{M}$ was defined as the quotient filtration on $\mathcal{D}_{\mathfrak{g} \times \mathfrak{t}}/\mathcal{I}$.

5. A GENERALIZATION OF A CONSTRUCTION OF BEILINSON AND KAZHDAN

5.1. The universal stabilizer sheaf. Given a G -action on an irreducible scheme X , one defines the “universal stabilizer” scheme \mathcal{G}_X as a scheme theoretic preimage of the diagonal in $X \times X$ under the morphism $G \times X \rightarrow X \times X$, $(g, x) \mapsto (gx, x)$. Set theoretically, one has $\mathcal{G}_X = \{(g, x) \in G \times X \mid gx = x\}$. The group G acts naturally on \mathcal{G}_X by $g_1 : (g, x) \mapsto (g_1 g g_1^{-1}, g_1 x)$. The second projection $\mathcal{G}_X \rightarrow X$, $(g, x) \mapsto x$ gives \mathcal{G}_X the natural structure of a G -equivariant group scheme over X . The Lie algebra of the group scheme \mathcal{G}_X is a coherent sheaf $\mathfrak{g}_X := \text{Ker}(\mathfrak{g} \otimes \mathcal{O}_X \rightarrow \mathcal{T}_X)$, the kernel of the natural “infinitesimal action” morphism. Let $\mathfrak{g}_X^* := \text{Hom}_{\mathcal{O}_X}(\mathfrak{g}_X, \mathcal{O}_X)$ be the (nonderived) dual of \mathfrak{g}_X .

Let L be a finite dimensional \mathfrak{g} -representation. There are natural morphisms of sheaves

$$L \otimes \mathcal{O}_X \longrightarrow \text{Hom}_{\mathbb{C}}(\mathfrak{g}, L) \otimes \mathcal{O}_X = L \otimes \mathfrak{g}^* \otimes \mathcal{O}_X \longrightarrow L \otimes \mathfrak{g}_X^*. \quad (5.1.1)$$

Here, the first morphism is induced by the linear map $L \rightarrow \text{Hom}_{\mathbb{C}}(\mathfrak{g}, L)$ resulting from the \mathfrak{g} -action on L , and the second morphism is induced by the sheaf imbedding $\mathfrak{g}_X \rightarrow \mathfrak{g} \otimes \mathcal{O}_X$.

Let $L^{\mathfrak{g}^X}$ denote the kernel of the composite morphism in (5.1.1). Thus, $L^{\mathfrak{g}^X}$ is a G -equivariant coherent subsheaf of $L \otimes \mathcal{O}_X$. The geometric fiber of the sheaf $L^{\mathfrak{g}^X}$ at a sufficiently general point $x \in X$ equals $L^{\mathfrak{g}_x}$.

Lemma 5.1.2. *For an irreducible normal variety X with a G -action, we have*

(i) *The sheaf imbedding $L^{\mathfrak{g}^X} \rightarrow L \otimes \mathcal{O}_X$ induces an isomorphism $\Gamma(X, L^{\mathfrak{g}^X})^G \xrightarrow{\sim} (L \otimes \mathbb{C}[X])^G$, provided general points of X have connected stabilizers.*

(ii) *Let $j : U \hookrightarrow X$ be an imbedding of a Zariski open subset such that $\dim(X \setminus U) \leq \dim X - 2$. Then, the canonical morphism $L^{\mathfrak{g}^X} \rightarrow j_* j^*(L^{\mathfrak{g}^X})$, resp. $\mathfrak{g}_X \rightarrow j_* j^*(\mathfrak{g}_X)$, is an isomorphism.*

Proof. For any $x \in X$ and any G -equivariant morphism $f : X \rightarrow L$, the element $f(x) \in L$ is fixed by the group G_x . This implies (i).

To prove (ii), we use the isomorphism $\mathfrak{g} \otimes \mathcal{O}_X \xrightarrow{\sim} j_* j^*(\mathfrak{g} \otimes \mathcal{O}_X)$, due to normality of X . Thus, we deduce

$$\mathfrak{g}_X = \text{Ker}[j_*(\mathfrak{g} \otimes \mathcal{O}_U) \rightarrow \mathcal{T}_X] = j_*(\text{Ker}[\mathfrak{g} \otimes \mathcal{O}_U \rightarrow j^* \mathcal{T}_X]) = j_* \mathfrak{g}_U.$$

Next, let $V \subset X$ be an open subset and let $s \in \Gamma(V, j_* j^*(L^{\mathfrak{g}^X}))$. One may view s as a morphism $V \cap U \rightarrow L$. This morphism can be extended to a regular map $\bar{s} : V \rightarrow L$, since X is normal. Let s' be the image of \bar{s} under the composite morphism in (5.1.1). Thus, $s' \in \Gamma(V, L \otimes \mathfrak{g}_X^*)$ and, by the definition of the sheaf $L^{\mathfrak{g}^X}$, we have $s'|_U = 0$.

Now, the dual of a coherent sheaf is a torsion free sheaf. Therefore, the sheaf $L \otimes \mathfrak{g}_X^*$ is torsion free. Hence, $s'|_{V \cap U} = 0$ implies that $s' = 0$. We deduce that \bar{s} is actually a section (over V) of the sheaf $L^{\mathfrak{g}^X}$, the kernel of the map (5.1.1). Since $\bar{s}|_U = s$, we have proved that the morphism $L^{\mathfrak{g}^X} \rightarrow j_* j^*(L^{\mathfrak{g}^X})$ is surjective. This morphism is injective since $L^{\mathfrak{g}^X}$, being a subsheaf of a locally free sheaf $L \otimes \mathcal{O}_X$, is a torsion free sheaf. \square

5.2. A canonical filtration. Let \mathbb{X} be the weight lattice of \mathfrak{g} . Given a Borel subalgebra \mathfrak{b} , we identify elements of \mathbb{X} with linear functions on $\mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]$. We let R (= the set of roots), resp. R^+ (= the set of positive roots) be the set of nonzero weights of the ad \mathfrak{b} -action on \mathfrak{g} , resp. on $\mathfrak{g}/\mathfrak{b}$. We introduce a partial order on \mathbb{X} by setting $\lambda' \preceq \lambda$ if $\lambda - \lambda'$ is a sum of positive roots.

Fix a finite dimensional \mathfrak{g} -module L and a Borel subalgebra \mathfrak{b} . Let $L = \bigoplus_{\lambda \in \mathbb{X}} L_t^\lambda$ be the weight decomposition of L with respect to a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{b}$. For any $\lambda \in \mathbb{X}$, we put $L_t^{\preceq \lambda} := \bigoplus_{\lambda' \preceq \lambda} L_t^{\lambda'}$, resp. $L_t^{\prec \lambda} := \bigoplus_{\lambda' \preceq \lambda, \lambda' \neq \lambda} L_t^{\lambda'}$. This is a \mathfrak{b} -stable subspace of L that does not depend on the choice of a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{b}$. Therefore, the above construction associates with each Borel subalgebra \mathfrak{b} a canonically defined \mathfrak{b} -stable filtration, to be denoted by $L_{\mathfrak{b}}^{\preceq \bullet}$, labeled by the partially ordered set \mathbb{X} .

One may let the Borel subalgebra \mathfrak{b} vary inside the flag variety. The family $\{L_{\mathfrak{b}}^{\preceq \bullet}, \mathfrak{b} \in \mathcal{B}\}$, of filtrations on L , then gives a filtration $L_{\mathcal{B}}^{\preceq \bullet}$ of the trivial sheaf $L \otimes \mathcal{O}_{\mathcal{B}}$ by G -equivariant locally free $\mathcal{O}_{\mathcal{B}}$ -subsheaves. We put $L_{\mathcal{B}}^\lambda = L_{\mathcal{B}}^{\preceq \lambda} / L_{\mathcal{B}}^{\prec \lambda}$ and let $L_{\mathcal{B}} := \bigoplus_{\lambda \in \mathbb{X}} L_{\mathcal{B}}^\lambda$, an associated graded sheaf. This is a G -equivariant locally free sheaf on \mathcal{B} .

In the special case $L = \mathfrak{g}$, the adjoint representation, one has $L_{\mathfrak{b}}^{\preceq 0} = \mathfrak{b}$, resp. $L_{\mathfrak{b}}^{\prec 0} = [\mathfrak{b}, \mathfrak{b}]$, for any $\mathfrak{b} \in \mathcal{B}$. Furthermore, the above construction yields, for each root $\alpha \in R$, a G -equivariant line bundle $\mathfrak{g}_{\mathcal{B}}^\alpha := \mathfrak{g}_{\mathcal{B}}^{\preceq \alpha} / \mathfrak{g}_{\mathcal{B}}^{\prec \alpha}$. Thus, we have

$$\mathfrak{g}_{\mathcal{B}} := \mathfrak{g}_{\mathcal{B}}^0 \oplus \left(\bigoplus_{\alpha \in R} \mathfrak{g}_{\mathcal{B}}^\alpha \right). \quad (5.2.1)$$

The Lie bracket on \mathfrak{g} induces a natural fiberwise bracket on the locally free sheaf $\mathfrak{g}_{\mathcal{B}}$. Further, it is easy to check that the invariant bilinear form on \mathfrak{g} induces an isomorphism $\mathfrak{g}_{\mathcal{B}}^{-\alpha} \cong (\mathfrak{g}_{\mathcal{B}}^\alpha)^*$, of line bundles on \mathcal{B} .

Given a finite dimensional representation L , the action map $\mathfrak{g} \otimes L \rightarrow L$ induces, for any $\alpha \in R$ and $\lambda \in \mathbb{X}$, a well defined morphism $\mathfrak{g}_{\mathcal{B}}^{\alpha} \otimes L_{\mathcal{B}}^{\lambda} \rightarrow L_{\mathcal{B}}^{\lambda+\alpha}$, of G -equivariant locally free sheaves on \mathcal{B} . This way, one obtains a natural ‘action’ morphism $\mathfrak{g}_{\mathcal{B}} \otimes L_{\mathcal{B}} \rightarrow L_{\mathcal{B}}$.

Note that any Borel subgroup $B \subset G$ acts trivially on the vector space $L_{\mathfrak{b}}^{\leq 0}/L_{\mathfrak{b}}^{< 0}$. It follows that there is a canonical isomorphism $L_{\mathfrak{b}}^{\leq 0}/L_{\mathfrak{b}}^{< 0} = L_{\mathfrak{b}'}^{\leq 0}/L_{\mathfrak{b}'}^{< 0}$, for any Borel subalgebras \mathfrak{b} and \mathfrak{b}' . In other words, the $\mathcal{O}_{\mathcal{B}}$ -sheaf $L_{\mathcal{B}}^0$ comes equipped with a canonical G -equivariant trivialization. In the case of the adjoint representation the canonical trivialization reads $\mathfrak{g}_{\mathcal{B}}^0 = \mathfrak{h} \otimes \mathcal{O}_{\mathcal{B}}$, where the vector space \mathfrak{h} is referred to as the *universal Cartan algebra* of \mathfrak{g} . More generally, for an arbitrary representation L , one has the canonical G -equivariant isomorphism $L_{\mathcal{B}}^0 \cong L^{\mathfrak{h}} \otimes \mathcal{O}_{\mathcal{B}}$, where the vector space $L^{\mathfrak{h}}$ may be called the ‘*universal zero weight space*’ of L . The group G acts on $L^{\mathfrak{h}} \otimes \mathcal{O}_{\mathcal{B}}$ through its action on $\mathcal{O}_{\mathcal{B}}$.

Remark 5.2.2. In the above discussion, we have implicitly viewed elements of the weight lattice \mathbb{X} as linear functions on \mathfrak{h} , the universal Cartan algebra.

5.3. W -action. Let $\tilde{\mathfrak{g}}^r = \mu^{-1}(\mathfrak{g}^r)$, a Zariski open subset of $\tilde{\mathfrak{g}}$. An important role below will be played by the following commutative diagram

$$\begin{array}{ccccc} \tilde{\mathfrak{g}}^r & \xrightarrow[\sim]{\pi} & \mathfrak{r} = \mathfrak{g}^r \times_{\mathfrak{t}/W} \mathfrak{t} & \xrightarrow{\tilde{\gamma}} & \mathfrak{t} \\ & \searrow \mu & \downarrow pr & & \downarrow \vartheta \\ & & \mathfrak{g}^r & \xrightarrow{\gamma} & \mathfrak{g}^r/G = \mathfrak{t}/W, \end{array} \quad (5.3.1)$$

In this diagram, the map pr is the first projection, ϑ is the quotient map, $\tilde{\gamma}$ is the second projection and γ is the adjoint quotient morphism. The map π , in the diagram, is the isomorphism from Proposition 3.1.1(i). Note that the square on the right of diagram (5.3.1) is cartesian, by definition.

Given a \mathfrak{g} -representation L , we pull-back the canonical filtration on the trivial sheaf $L \otimes \mathcal{O}_{\mathcal{B}}$ via the vector bundle projection $\tilde{\mathfrak{g}} \rightarrow \mathcal{B}$. We obtain a filtration $L_{\tilde{\mathfrak{g}}}^{\leq \bullet}$, of the sheaf $L \otimes \mathcal{O}_{\tilde{\mathfrak{g}}}$, by G -equivariant subsheaves. The sheaves $L_{\tilde{\mathfrak{g}}}^{\lambda} := L_{\tilde{\mathfrak{g}}}^{\leq \lambda}/L_{\tilde{\mathfrak{g}}}^{< \lambda}$ are locally free, and so is $L_{\tilde{\mathfrak{g}}} = \bigoplus_{\lambda \in \mathbb{X}} L_{\tilde{\mathfrak{g}}}^{\lambda}$, an associated graded sheaf. In particular, for $\lambda = 0$ and $L = \mathfrak{g}$, the adjoint representation, one gets the sheaf $\mathfrak{g}_{\tilde{\mathfrak{g}}}^0 = \mathfrak{h} \otimes \mathcal{O}_{\tilde{\mathfrak{g}}}$.

The Weyl group W acts on the fibers of the map pr in (5.3.1). We may transport this action via the isomorphism $\tilde{\mathfrak{g}}^r \xrightarrow{\sim} \mathfrak{r}$ to get a W -action on $\tilde{\mathfrak{g}}^r$. Thus, for any $w \in W$ and $(\mathfrak{b}, x) \in \tilde{\mathfrak{g}}^r$ there is a unique Borel subalgebra \mathfrak{b}' such that we have $w(\mathfrak{b}, x) = (\mathfrak{b}', x)$.

Assume now that the element x is regular semisimple. Then, \mathfrak{g}_x is a Cartan subalgebra of \mathfrak{g} contained in $\mathfrak{b} \cap \mathfrak{b}'$. Thus, one has the following chain of isomorphisms

$$\mathfrak{h} = \mathfrak{g}_{\tilde{\mathfrak{g}}}^0|_{(\mathfrak{b}, x)} = \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}] \xleftarrow{\sim} \mathfrak{g}_x \xrightarrow{\sim} \mathfrak{b}'/[\mathfrak{b}', \mathfrak{b}'] = \mathfrak{g}_{\tilde{\mathfrak{g}}}^0|_{w(\mathfrak{b}, x)} = \mathfrak{h}. \quad (5.3.2)$$

Here, the composite map from the copy of \mathfrak{h} on the left to the copy of \mathfrak{h} on the right is the map $\mathfrak{h} \rightarrow \mathfrak{h}$, $h \mapsto w(h)$. One also has the dual map $\mathfrak{h}^* \rightarrow \mathfrak{h}^*$, $\lambda \mapsto w(\lambda)$. It follows that, given $\lambda \in \mathbb{X}$, there is an analogue of diagram (5.3.2) for any \mathfrak{g} -representation L ; it reads:

$$L_{\mathcal{B}}^{\lambda}|_{(\mathfrak{b}, x)} = L_{\mathfrak{b}}^{\leq \lambda}/L_{\mathfrak{b}}^{< \lambda} \xleftarrow{\sim} L_{\mathfrak{g}_x}^{\lambda} \xrightarrow{\sim} L_{\mathfrak{b}'}^{\leq w(\lambda)}/L_{\mathfrak{b}'}^{< w(\lambda)} = L_{\tilde{\mathfrak{g}}}^{w(\lambda)}|_{w(\mathfrak{b}, x)}.$$

Now, we let the Borel subalgebra \mathfrak{b} vary. We conclude that the above construction yields a canonical isomorphism

$$w^*(L_{\tilde{\mathfrak{g}}}^{\lambda})|_{\tilde{\mathfrak{g}}^{rs}} \cong L_{\tilde{\mathfrak{g}}}^{w(\lambda)}|_{\tilde{\mathfrak{g}}^{rs}}, \quad \forall \lambda \in \mathbb{X}, w \in W, \quad (5.3.3)$$

of G -equivariant locally free sheaves on $\tilde{\mathfrak{g}}^{rs} := \mu^{-1}(\mathfrak{g}^{rs})$:

In the special case where $\lambda = 0$, we have that $L_{\mathfrak{g}}^0 = L^{\mathfrak{h}} \otimes \mathcal{O}_{\tilde{\mathfrak{g}}}$ is a trivial sheaf. Hence, $w^*(L_{\tilde{\mathfrak{g}}^r}^0) = L_{\tilde{\mathfrak{g}}^r}^0$, canonically. Thus, the isomorphism in (5.3.3) yields a *canonical* W -action on the universal zero weight space $L^{\mathfrak{h}}$.

5.4. The morphisms λ^k and λ^L . In this section, we are interested in the universal stabilizer contruction of §5.1 in the special case where the group G acts on $X = \mathfrak{g}^r$ by the adjoint action. Below, we will use simplified notation $\mathbf{g} := \mathbf{g}_{\mathfrak{g}^r}$. Thus, \mathbf{g} is a rank r locally free sheaf on \mathfrak{g}^r . The geometric fiber of the sheaf \mathbf{g} at any point $x \in \mathfrak{g}^r$ equals \mathfrak{g}_x , the centralizer of x .

From now on, we let L be a finite dimensional rational G -representation such that the set of weights of L is *contained in the root lattice*. In this case, the results of Kostant [Ko] insure that $L^{\mathbf{g}}$ is a locally free sheaf on \mathfrak{g}^r .

Let $\mu^*(L^{\mathbf{g}})$ be the pullback of $L^{\mathbf{g}}$ via the map $\mu : \tilde{\mathfrak{g}}^r \rightarrow \mathfrak{g}^r$. Thus, both $\mu^*(L^{\mathbf{g}})$ and $L_{\tilde{\mathfrak{g}}^r}^{\leq 0}$ are G -equivariant locally free subsheaves of the trivial sheaf $L \otimes \mathcal{O}_{\tilde{\mathfrak{g}}^r}$.

Lemma 5.4.1. *The sheaf $\mu^*(L^{\mathbf{g}})$ is a subsheaf of $L_{\tilde{\mathfrak{g}}^r}^{\leq 0}$ so, for any Borel subalgebra \mathfrak{b} and any $x \in \mathfrak{g}^r \cap \mathfrak{b}$, we have an inclusion $L^{\mathfrak{g}_x} \subset L_{\mathfrak{b}}^{\leq 0}$.*

Proof. We may (and will) identify the locally free sheaf $\mu^*(L^{\mathbf{g}})$, resp. $L_{\tilde{\mathfrak{g}}^r}^{\leq 0}$, with the corresponding vector bundle on $\tilde{\mathfrak{g}}^r$. Thus, both $\mu^*(L^{\mathbf{g}})$ and $L_{\tilde{\mathfrak{g}}^r}^{\leq 0}$ are G -equivariant sub vector bundles of the trivial vector bundle on $\tilde{\mathfrak{g}}^r$ with fiber L .

Let $(\mathfrak{b}, x) \in \tilde{\mathfrak{g}}^{rs}$. Then, \mathfrak{g}_x is a Cartan subalgebra contained in \mathfrak{b} and, by definition, we have $L^{\mathfrak{g}_x} = L_{\mathfrak{g}_x}^0 \subset L_{\mathfrak{b}}^{\leq 0}$. Hence, for any $(\mathfrak{b}, x) \in \tilde{\mathfrak{g}}^{rs}$, the fiber of the vector bundle $\mu^*(L^{\mathbf{g}})$ at the point (\mathfrak{b}, x) is contained in the corresponding fiber of the vector bundle $L_{\tilde{\mathfrak{g}}^r}^{\leq 0}$. The statement of the lemma follows from this by continuity since the set $\tilde{\mathfrak{g}}^{rs}$ is dense in $\tilde{\mathfrak{g}}^r$. \square

Thanks to the above lemma, one has the following chain of canonical morphisms of locally free G -equivariant sheaves on $\tilde{\mathfrak{g}}^r$:

$$\mu^*(L^{\mathbf{g}}) \longrightarrow L_{\tilde{\mathfrak{g}}^r}^{\leq 0} \longrightarrow L_{\tilde{\mathfrak{g}}^r}^{\leq 0} / L_{\tilde{\mathfrak{g}}^r}^{\leq 0} = L_{\tilde{\mathfrak{g}}^r}^0 = L^{\mathfrak{h}} \otimes \mathcal{O}_{\tilde{\mathfrak{g}}^r}. \quad (5.4.2)$$

We are now going to transport various sheaves on $\tilde{\mathfrak{g}}^r$ to \mathfrak{r} via the isomorphism $\pi : \tilde{\mathfrak{g}}^r \xrightarrow{\sim} \mathfrak{r}$ of diagram (5.3.1). This way, one obtains a filtration $L_{\mathfrak{r}}^{\bullet}$ of the trivial sheaf $L \otimes \mathcal{O}_{\mathfrak{r}}$ and an associated graded sheaf $L_{\mathfrak{r}} = \bigoplus_{\lambda \in \mathbb{X}} L_{\mathfrak{r}}^{\lambda}$. Further, we factor the morphism μ in (5.3.1) as a composition $\mu = pr \circ \pi$. Thus, we have $\mu^*(L^{\mathbf{g}}) = \pi^* pr^*(L^{\mathbf{g}})$, where we put $L_{\mathfrak{r}}^{\mathbf{g}} := pr^*(L^{\mathbf{g}})$. Transporting the composite morphism in (5.4.2) via the isomorphism π yields a morphism

$$pr^*(L^{\mathbf{g}}) \longrightarrow L_{\mathfrak{r}}^0 = L^{\mathfrak{h}} \otimes \mathcal{O}_{\mathfrak{r}}. \quad (5.4.3)$$

Applying adjunction to the above morphism, one obtains a morphism $\lambda^L : L^{\mathbf{g}} \rightarrow L^{\mathfrak{h}} \otimes pr_* \mathcal{O}_{\mathfrak{r}}$, of locally free G -equivariant sheaves on \mathfrak{g}^r .

An important special case of the above setting is the case where $L = \mathfrak{g}$, the adjoint representation. The weights of the adjoint representation are clearly contained in the root lattice. Furthermore, for $L = \mathfrak{g}$ one has $L^{\mathfrak{g}_x} = \mathfrak{g}_x$, hence, we have $L^{\mathbf{g}} = \mathbf{g}$. Since $\mathfrak{g}_{\mathfrak{b}}^{\leq 0} = \mathfrak{b}$, we see that Lemma 5.4.1 reduces in the case $L = \mathfrak{g}$ to a well known result saying that, for any $x \in \mathfrak{b} \cap \mathfrak{g}^r$, one has an inclusion $\mathfrak{g}_x \subset \mathfrak{b}$. Further, the composite morphism in (5.4.2) becomes a morphism $pr^* \mathbf{g} \rightarrow \mathfrak{h} \otimes \mathcal{O}_{\mathfrak{r}}$. We take exterior powers of that morphism. This gives, for each $k \geq 0$, an induced morphism $pr^*(\wedge^k \mathbf{g}) \rightarrow \wedge^k \mathfrak{h} \otimes \mathcal{O}_{\mathfrak{r}}$. Finally, applying adjunction one obtains a morphism $\lambda^k : \wedge^k \mathbf{g} \rightarrow \wedge^k \mathfrak{h} \otimes pr_* \mathcal{O}_{\mathfrak{r}}$.

Associated with any W -module E , there is a G -equivariant coherent sheaf $(E \otimes pr_* \mathcal{O}_t)^W$ on \mathfrak{g}^r . In particular, one may let $E = L^{\mathfrak{h}}$, the universal zero weight space of a G -module L

Lemma 5.4.4. (i) *The sheaf $(E \otimes pr_* \mathcal{O}_t)^W$ is locally free for any W -module E .*

(ii) *The image of the morphism λ^L , resp. λ^k , $k \geq 0$, is contained in the corresponding sheaf of W -invariants, so the morphism in question factors through an injective morphism*

$$\lambda^L : L^{\mathfrak{g}} \rightarrow (L^{\mathfrak{h}} \otimes pr_* \mathcal{O}_t)^W, \quad \text{resp.} \quad \lambda^k : \wedge^k \mathfrak{g} \rightarrow (\wedge^k \mathfrak{h} \otimes pr_* \mathcal{O}_t)^W, \quad k \geq 0. \quad (5.4.5)$$

(iii) *On the open dense set $\mathfrak{g}^{rs} \subset \mathfrak{g}^r$, each of the morphisms in (5.4.5) is an isomorphism.*

Sketch of Proof. The statement of part (i) is well known but we recall the proof, for completeness. First, we apply base change for the cartesian square in diagram (5.3.1). This yields a chain of isomorphisms $pr_* \mathcal{O}_t = pr_* \tilde{\gamma}^* \mathcal{O}_t = \gamma^* \vartheta_* \mathcal{O}_t$. Here, the sheaf on the right is locally free and, moreover, the Weyl group W acts on the geometric fibers of that sheaf via the regular representation. We deduce that the sheaf $pr_* \mathcal{O}_t$ has similar properties. Part (i) follows from this.

The proof of part (iii) is straightforward. Finally, part (ii) follows from part (iii) ‘by continuity’, using that the sheaves $L^{\mathfrak{g}}$ and $\wedge^k \mathfrak{g}$ are locally free. \square

The theorem below, which is the main result of this section, is inspired by an idea due to Beilinson and Kazhdan. In [BK], the authors considered the isomorphism of part (i) of Theorem 5.4.6 in the special case $k = 1$ (equivalently, of part (ii) in the case $L = \mathfrak{g}$).

Theorem 5.4.6. (i) *For any $k \geq 1$, the morphism $\lambda^k : \wedge^k \mathfrak{g} \xrightarrow{\sim} (\wedge^k \mathfrak{h} \otimes pr_* \mathcal{O}_t)^W$ is an isomorphism.*

(ii) *For any small representation L , the morphism $\lambda^L : L^{\mathfrak{g}} \xrightarrow{\sim} (L^{\mathfrak{h}} \otimes pr_* \mathcal{O}_t)^W$ is an isomorphism.*

The proof of Theorem 5.4.6 will be completed in §5.6.

In §5.7 we sketch a generalization of part (ii) of the above theorem to the case of an arbitrary, not necessarily small, representation L .

5.5. Proof of Theorem 5.4.6(ii). Let $\text{Coh}^G(\mathfrak{g}^r)$ be the category of G -equivariant coherent sheaves on \mathfrak{g}^r . We begin with an easy

Lemma 5.5.1. *The functor $\Gamma : \text{Coh}^G(\mathfrak{g}^r) \xrightarrow{\sim} \mathbb{C}[t]^W\text{-mod}$, $\mathcal{F} \mapsto \Gamma(\mathfrak{g}^r, \mathcal{F})^G$ is an equivalence.*

Proof. Recall the adjoint quotient morphism γ , cf. (5.3.1). We have a diagram of functors

$$\text{Coh}^G(\mathfrak{g}^r) \begin{array}{c} \xrightarrow{\mathcal{F} \mapsto (\gamma_* \mathcal{F})^G} \\ \xleftarrow{\gamma^*} \end{array} \text{Coh}(\mathfrak{t}/W) \xrightarrow{\Gamma(\mathfrak{t}/W, -)} \mathbb{C}[t]^W\text{-mod}.$$

It is known, thanks to results of Kostant [Ko], that γ is a smooth morphism, moreover, each fiber of that morphism is a single G -orbit. It follows by equivariant descent that the functor $\mathcal{F} \mapsto (\gamma_* \mathcal{F})^G$ is an equivalence, with γ^* being its quasi-inverse. The functor $\Gamma(\mathfrak{t}/W, -)$, in the above diagram, is an equivalence since the variety \mathfrak{t}/W is affine. Finally, the functor Γ is isomorphic to the composite functor $\mathcal{F} \mapsto \Gamma(\mathfrak{t}/W, (\gamma_* \mathcal{F})^G) = \Gamma(\mathfrak{g}^r, \mathcal{F})^G$. It follows that Γ is also an equivalence. \square

It is useful to transport the morphisms in (5.4.5) via the equivalence Γ of Lemma 5.5.1. To this end, for a G -module L , we compute

$$\Gamma(L^{\mathfrak{g}}) = \Gamma(\mathfrak{g}^r, L^{\mathfrak{g}})^G = \Gamma(\mathfrak{g}, L^{\mathfrak{g}_0}) = (L \otimes \mathbb{C}[\mathfrak{g}])^G, \quad (5.5.2)$$

where the second equality holds by part (ii) of Lemma 5.1.2 and the third equality holds by part (i) of the same lemma.

Recall next that \mathfrak{r} is known to be a normal variety and the set $\mathfrak{r} \setminus \mathfrak{t}$ has codimension ≥ 2 in \mathfrak{r} . This yields natural isomorphisms $\mathbb{C}[\mathfrak{g}] \otimes_{\mathbb{C}[\mathfrak{t}]^W} \mathbb{C}[\mathfrak{t}] = \mathbb{C}[\mathfrak{r}] \xrightarrow{\sim} \mathbb{C}[\mathfrak{r}]$. In particular, we deduce $\mathbb{C}[\mathfrak{r}]^G = (\mathbb{C}[\mathfrak{g}] \otimes_{\mathbb{C}[\mathfrak{t}]^W} \mathbb{C}[\mathfrak{t}])^G = \mathbb{C}[\mathfrak{t}]$.

Thus, for any W -representation E , we find

$$\begin{aligned} \Gamma((E \otimes pr_* \mathcal{O}_{\mathfrak{r}})^W) &= \Gamma(\mathfrak{g}^r, (E \otimes pr_* \mathcal{O}_{\mathfrak{r}})^W)^G = (E \otimes \Gamma(\mathfrak{g}^r, pr_* \mathcal{O}_{\mathfrak{r}}))^{W \times G} \\ &= (E \otimes \Gamma(\mathfrak{r}, \mathcal{O}_{\mathfrak{r}}))^{W \times G} = (E \otimes \mathbb{C}[\mathfrak{r}]^G)^W = (E \otimes \mathbb{C}[\mathfrak{t}])^W. \end{aligned} \quad (5.5.3)$$

Now, fix $\mathfrak{t} \subset \mathfrak{b} \subset \mathfrak{g}$ and write $i : \mathfrak{t} \hookrightarrow \mathfrak{g}$ for the resulting imbedding of the Cartan subalgebra. This gives, for any G -representation L , an identification $L^{\mathfrak{t}} \xrightarrow{\sim} L^{\mathfrak{b}}$ and also a natural restriction map $i^* : (L \otimes \mathbb{C}[\mathfrak{g}])^G \rightarrow (L^{\mathfrak{t}} \otimes \mathbb{C}[\mathfrak{t}])^W$.

Lemma 5.5.4. *The following diagram commutes*

$$\begin{array}{ccc} \Gamma(L^{\mathfrak{g}}) & \xrightarrow{(5.5.2)} & (L \otimes \mathbb{C}[\mathfrak{g}])^G \\ \downarrow \Gamma(\lambda^L) & & \downarrow i^* \\ \Gamma((L^{\mathfrak{b}} \otimes pr_* \mathcal{O}_{\mathfrak{r}})^W) & \xrightarrow{(5.5.3)} & (L^{\mathfrak{b}} \otimes \mathbb{C}[\mathfrak{t}])^W = (L^{\mathfrak{t}} \otimes \mathbb{C}[\mathfrak{t}])^W \end{array} \quad (5.5.5)$$

Proof. To prove commutativity of the diagram it suffices to check this after localization to the open dense set $\mathfrak{g}^{rs} \subset \mathfrak{g}^r$ of regular semisimple elements. The latter is straightforward and is left for the reader. \square

By commutativity of diagram (5.5.5), using the equivalence of Lemma 5.5.1 we conclude that the morphism λ^L , in (5.4.5), is an isomorphism if and only if so is the map i^* on the right of diagram (5.5.5). At this point, the proof of Theorem 5.4.6(ii) is completed by the following result of B. Broer [Br].

Proposition 5.5.6. *The restriction map $i^* : (L \otimes \mathbb{C}[\mathfrak{g}])^G \rightarrow (L^{\mathfrak{t}} \otimes \mathbb{C}[\mathfrak{t}])^W$ is an isomorphism, for any small representation L .* \square

5.6. Proof of Theorem 5.4.6(i). Write Ω_X^k for the sheaf of k -forms on a smooth variety X . Thus, using the identification $\mathfrak{g}^* \cong \mathfrak{g}$, resp. $\mathfrak{t}^* \cong \mathfrak{t}$, for each $k \geq 0$, we have an isomorphism $\wedge^k \mathfrak{g} \otimes \mathcal{O}_{\mathfrak{g}} \cong \Omega_{\mathfrak{g}}^k$, resp. $\wedge^k \mathfrak{t} \otimes \mathcal{O}_{\mathfrak{t}} \cong \Omega_{\mathfrak{t}}^k$.

Let f_1, \dots, f_r be a set of homogeneous generators of $\mathbb{C}[\mathfrak{g}]^G$, a free polynomial algebra. Thus, one may think of the 1-forms df_i , $i = 1, \dots, r$, as being sections of the sheaf $\mathfrak{g} \otimes \mathcal{O}_{\mathfrak{g}} \cong \Omega_{\mathfrak{g}}^1$. A result of Kostant says that the values $(df_1)|_x, \dots, (df_r)|_x$, of these sections at any point $x \in \mathfrak{g}^r$, give a basis of the vector space $\mathfrak{g}_x = \mathfrak{g}|_x$. Hence, the r -tuple (df_1, \dots, df_r) provides a basis of sections of the sheaf \mathfrak{g} .

Next, let \bar{f}_i be the image of f_i under the Chevalley isomorphism $\mathbb{C}[\mathfrak{g}]^G \xrightarrow{\sim} \mathbb{C}[\mathfrak{t}/W]$. The map $h \mapsto (\bar{f}_1(h), \dots, \bar{f}_r(h))$ yields an isomorphism $\mathfrak{t}/W \xrightarrow{\sim} \mathbb{C}^r$. Hence, the r -tuple $d\bar{f}_1, \dots, d\bar{f}_r$, of 1-forms on \mathfrak{t}/W , provides a basis of sections of the sheaf $\Omega_{\mathfrak{t}/W}^1$. Therefore, the r -tuple $\gamma^*(d\bar{f}_1), \dots, \gamma^*(d\bar{f}_r)$ provides a basis of sections of the sheaf $\gamma^* \Omega_{\mathfrak{t}/W}^1$, on \mathfrak{g}^r .

Thus, the assignment $\gamma^*(d\bar{f}_i) \mapsto df_i$, $i = 1, \dots, r$, gives an isomorphism of sheaves $\gamma^* \Omega_{\mathfrak{t}/W}^1 \xrightarrow{\sim} \mathfrak{g}$. Taking exterior powers, one obtains, for each $k \geq 0$, an isomorphism $\Psi : \gamma^* \Omega_{\mathfrak{t}/W}^k \xrightarrow{\sim} \wedge^k \mathfrak{g}$, of locally free sheaves on \mathfrak{g}^r .

On the other hand, we recall that the square on the right of diagram (5.3.1) is cartesian and the morphism ϑ in the diagram is finite and flat. Hence, by flat base change, for any

$k \geq 0$, we get W -equivariant isomorphisms $pr_*\mathcal{O}_\tau = pr_*\tilde{\gamma}^*\mathcal{O}_\mathfrak{t} = \gamma^*\vartheta_*\mathcal{O}_\mathfrak{t}$. Tensoring here each term by the W -module $\wedge^k \mathfrak{t}$ and taking W -invariants in the resulting sheaves, yields a natural sheaf isomorphism $\Phi : (\wedge^k \mathfrak{t} \otimes pr_*\mathcal{O}_\tau)^W \xrightarrow{\sim} \gamma^*[(\vartheta_*(\wedge^k \mathfrak{t} \otimes \mathcal{O}_\mathfrak{t}))^W] = \gamma^*(\vartheta_*\Omega_\mathfrak{t}^k)^W$.

Combining all the above morphisms together, we obtain the following diagram

$$\gamma^*\Omega_{\mathfrak{t}/W}^k \xrightarrow[\sim]{\Psi} \wedge^k \mathfrak{g} \xrightarrow[(5.4.5)]{\lambda^k} (\wedge^k \mathfrak{t} \otimes pr_*\mathcal{O}_\tau)^W \xrightarrow[\sim]{\Phi} \gamma^*(\vartheta_*\Omega_\mathfrak{t}^k)^W \quad (5.6.1)$$

Lemma 5.6.2. *The composite morphism in (5.6.1) equals the pull back via the map γ of the canonical morphism of sheaves $\Omega_{\mathfrak{t}/W}^k \rightarrow (\vartheta_*\Omega_\mathfrak{t}^k)^W$.*

Proof. The statement amounts to showing that, for each i , the composite morphism (5.6.1) sends the section df_i to the section $\gamma^*(d(\vartheta^*f_i))$, where $d(\vartheta^*f_i) \in \vartheta_*\Omega_\mathfrak{t}^1$ is viewed as a 1-form on \mathfrak{t} . Furthermore, it suffices to check this on the open dense set of regular semisimple elements where it is clear. \square

At this point, we apply a result due to L. Solomon [So] saying that the canonical morphism $\Omega_{\mathfrak{t}/W}^k \rightarrow (\vartheta_*\Omega_\mathfrak{t}^k)^W$ is in fact an isomorphism. It follows, thanks to Lemma 5.6.2, that the morphism λ^k in the middle of diagram (5.6.1) must be an isomorphism. This completes the proof of Theorem 5.4.6(i). \square

Remark 5.6.3. There is an alternative approach to the proof that the morphisms in (5.4.5) are isomorphisms as follows. First of all, on the open set of regular semisimple elements our claim amounts to Lemma 5.4.4(ii). Next, one verifies the result in the case $\mathfrak{g} = \mathfrak{sl}_2$. It follows that the result holds for any reductive Lie algebra of semisimple rank 1. Using this, one deduces that the morphisms of locally free sheaves in (5.4.5) are isomorphisms outside a codimension 2 subset. The result follows.

A similar strategy can also be used to obtain direct proofs of the above mentioned results of Broer and Solomon, respectively.

5.7. The case of a not necessarily small representation. The sheaf morphism $\lambda^L : L^\mathfrak{g} \rightarrow (L^\mathfrak{h} \otimes pr_*\mathcal{O}_\tau)^W$, in (5.4.5), may fail to be surjective in the case where L is a not necessarily small representation. Nonetheless, using a result by Khoroshkin, Nazarov, and Vinberg [KNV], we will give a description of the image of that morphism for an arbitrary representation L .

To this end, we first apply the construction of sections 5.3 and §5.4 in the special case of the adjoint representation $L = \mathfrak{g}$. Thus, starting from the sheaf $\mathfrak{g}_\mathcal{B}$ on \mathcal{B} , see (5.2.1), the construction produces a sheaf $\mathfrak{g}_\tau := \mathfrak{h} \otimes \mathcal{O}_\tau \oplus \left(\bigoplus_{\alpha \in R} \mathfrak{g}_\tau^\alpha \right)$. This is a G -equivariant locally free sheaf on τ that comes equipped with a fiberwise Lie bracket.

For any representation L , there is a natural action morphism $\mathfrak{g}_\tau \otimes L_\tau \rightarrow L_\tau$. Taking adjoints and using the isomorphism $\mathfrak{g}_\tau^{-\alpha} \cong (\mathfrak{g}_\tau^\alpha)^*$ yields, for any $\alpha \in R$ and $\lambda \in \mathbb{X}$, an induced coaction morphism $e_\alpha : L_\tau^\lambda \rightarrow L_\tau^{\lambda+\alpha} \otimes \mathfrak{g}_\tau^{-\alpha}$. Iterating the latter morphism k times we obtain morphisms

$$e_\alpha^k : L_\tau^\lambda \rightarrow L_\tau^{\lambda+k\cdot\alpha} \otimes \mathfrak{g}_\tau^{-k\cdot\alpha}, \quad k = 1, 2, \dots$$

Let $\ker \alpha \subset \mathfrak{t}$ be the root hyperplane corresponding to a root $\alpha \in R^+$. The inverse image of this hyperplane via the map $\tilde{\gamma} : \tau \rightarrow \mathfrak{t}$, cf. (5.3.1), is a smooth G -stable irreducible divisor $D_\alpha \subset \tau$. Given $k \geq 1$, we use the standard notation $L_\tau^{k\cdot\alpha} \otimes \mathfrak{g}_\tau^{-k\cdot\alpha}(-k \cdot D_\alpha)$ for a subsheaf of $L_\tau^{k\cdot\alpha} \otimes \mathfrak{g}_\tau^{-k\cdot\alpha}$ formed by the sections which have a k -th order zero at the divisor D_α .

Our description of the image of the morphism λ^L is provided by the following result inspired by [KNV].

Theorem 5.7.1. *Let L be a finite dimensional \mathfrak{g} -representation such that the weights of L are contained in the root lattice. Then the morphism $\lambda^L : L^{\mathfrak{g}} \rightarrow (L^{\mathfrak{h}} \otimes pr_* \mathcal{O}_{\mathfrak{t}})^W$, in (5.4.5), yields an isomorphism of $L^{\mathfrak{g}}$ with a subsheaf \mathcal{L} of the sheaf $(L^{\mathfrak{h}} \otimes pr_* \mathcal{O}_{\mathfrak{t}})^W$ defined as follows*

$$\mathcal{L} := \{s \in (L^{\mathfrak{h}} \otimes pr_* \mathcal{O}_{\mathfrak{t}})^W \mid e_{\alpha}^k(pr^*(s)) \in L_{\mathfrak{t}}^{k \cdot \alpha} \otimes \mathfrak{g}_{\mathfrak{t}}^{-k \cdot \alpha}(-k \cdot D_{\alpha}), \quad \forall \alpha \in R^+, k \geq 1\}.$$

Remark 5.7.2. It is not difficult to check that, in the case where L is a small representation, one has $\mathcal{L} = (L^{\mathfrak{h}} \otimes pr_* \mathcal{O}_{\mathfrak{t}})^W$. So, Theorem 5.7.1 reduces in this case to Theorem 5.4.6(ii). \diamond

The proof of Theorem 5.7.1 is parallel to the arguments used in §5.5. There is an analogue of commutative diagram (5.5.5). The space $(L^{\mathfrak{h}} \otimes \mathbb{C}[\mathfrak{t}])^W$ in that diagram is replaced by its subspace defined in [KNV, p.1169], resp. the space $\Gamma((L^{\mathfrak{h}} \otimes pr_* \mathcal{O}_{\mathfrak{t}})^W)$ is replaced by $\Gamma(\mathcal{L})$. The role of Broer's result from Proposition 5.5.6 is played by [KNV, Theorem 2].

Theorem 5.7.1 will not be used in the rest of the paper; so, details of the proof will be given elsewhere.

6. GEOMETRY OF THE COMMUTING SCHEME

6.1. Another isospectral variety. We write $x = h_x + n_x$ for the Jordan decomposition of an element $x \in \mathfrak{g}$. We say that a pair $(x, y) \in \mathfrak{G}$ is semisimple if both x and y are semisimple elements of \mathfrak{g} . Let $\mathfrak{G}^{\mathfrak{b}}$ be the set of pairs $(x, y) \in \mathfrak{G}$ such that there exists a Borel subalgebra that contains both x and y .

Lemma 6.1.1. (i) *Let $(x, y) \in \mathfrak{G}$. Then $(x, y) \in \mathfrak{G}^{\mathfrak{b}}$ if and only if there exists a semisimple pair $(h_1, h_2) \in \mathfrak{T}$ such that, for any polynomial $f \in \mathbb{C}[\mathfrak{G}]^G$, we have $f(x, y) = f(h_1, h_2)$.*

(ii) *If $(x, y) \in \mathfrak{G}$ then the G -diagonal orbit of the pair (h_x, h_y) is the unique closed G -orbit contained in the closure of the G -diagonal orbit of (x, y) .*

Proof. Let $\mathfrak{t} \subset \mathfrak{b}$ be Cartan and Borel subalgebras of \mathfrak{g} and let T be the maximal torus corresponding to \mathfrak{t} . Clearly, there exists a suitable one parameter subgroup $\gamma : \mathbb{C}^{\times} \rightarrow T$ such that, for any $t \in \mathfrak{t}$ and $n \in [\mathfrak{b}, \mathfrak{b}]$, one has $\lim_{z \rightarrow 0} \text{Ad } \gamma(z)(t + n) = t$.

To prove (i), let $(x, y) \in \mathfrak{b} \times \mathfrak{b}$. We can write $x = h_1 + n_1$, $y = h_2 + n_2$ where $h_i \in \mathfrak{t}$ and $n_i \in [\mathfrak{b}, \mathfrak{b}]$. We see from the above that the pair (h_1, h_2) is contained in the closure of the G -orbit of the pair (x, y) . Hence, for any $f \in \mathbb{C}[\mathfrak{G}]^G$, we have $f(x, y) = f(h_1, h_2)$.

Conversely, let $(h_1, h_2) \in \mathfrak{T}$ and let $(x, y) \in \mathfrak{G}$ be such that $f(x, y) = f(h_1, h_2)$ holds for any $f \in \mathbb{C}[\mathfrak{G}]^G$. The group G_{h_1, h_2} is reductive. Hence, the G -diagonal orbit of (h_1, h_2) is closed in \mathfrak{G} , cf. eg. [GOV]. Moreover, this G -orbit is the unique closed G -orbit contained in the closure of the G -orbit of the pair (x, y) since G -invariant polynomials on \mathfrak{G} separate closed G -orbits. By the Hilbert-Mumford criterion, we deduce that there exists a suitable one parameter subgroup $\gamma : \mathbb{C}^{\times} \rightarrow G$ such that, conjugating the pair (h_1, h_2) if necessary, one gets $\lim_{z \rightarrow 0} \text{Ad } \gamma(z)(x, y) = (h_1, h_2)$.

Now, let \mathfrak{a} be the Lie subalgebra of \mathfrak{g} generated by the elements x and y and let \mathfrak{t} be a Cartan subalgebra such that $\lim_{z \rightarrow 0} \text{Ad } \gamma(z)(x, y) \in \mathfrak{t} \times \mathfrak{t}$. We deduce from the above that one has $\lim_{z \rightarrow 0} \text{Ad } \gamma(z)([\mathfrak{a}, \mathfrak{a}]) \subset [\mathfrak{t}, \mathfrak{t}] = 0$. This implies that any element of $[\mathfrak{a}, \mathfrak{a}]$ is nilpotent. Hence, $[\mathfrak{a}, \mathfrak{a}]$ is a nilpotent Lie algebra, by Engel's theorem. We conclude that \mathfrak{a} is a solvable Lie algebra. Hence, there exists a Borel subalgebra \mathfrak{b} such that $\mathfrak{a} \subset \mathfrak{b}$ and (i) is proved.

To prove (ii), observe that the elements h_x, h_y, n_x, n_y generate an abelian Lie subalgebra of \mathfrak{g} . Hence, there exists a Borel subalgebra \mathfrak{b} that contains all of them. Choose a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{b}$ such that $h_x, h_y \in \mathfrak{t}$. Then, the argument at the beginning of the proof shows that the pair (h_x, h_y) is contained in the closure of the G -diagonal orbit of (x, y) . \square

Note that, by definition, we have $\mathfrak{G}^b = [\mu(\tilde{\mathfrak{G}})]_{\text{red}}$, the reduced image of the morphism $\mu : \tilde{\mathfrak{G}} \rightarrow \mathfrak{G}$, see §3.1. The following result implies Proposition 3.1.1(ii).

Proposition 6.1.2. *The morphism $\mu \times \nu : \tilde{\mathfrak{G}} \rightarrow [\mathfrak{G} \times_{\mathfrak{G}/G} \mathfrak{T}]_{\text{red}}$ is birational and proper.*

The first projection $\mathfrak{G} \times \mathfrak{T} \rightarrow \mathfrak{G}$ induces a birational finite morphism $[\mathfrak{G} \times_{\mathfrak{G}/G} \mathfrak{T}]_{\text{red}} \rightarrow \mathfrak{G}^b$.

Proof. Let \mathfrak{G}^\heartsuit be the set of pairs $(x, y) \in \mathfrak{G}$ such that the vector space $\mathbb{C}x + \mathbb{C}y \subset \mathfrak{g}$ spanned by x and y is 2-dimensional and, moreover, any nonzero element of that vector space is regular in \mathfrak{g} . It is clear that \mathfrak{G}^\heartsuit is a G -stable Zariski open and dense subset of \mathfrak{G} .

According to [CM], Lemma 6(i), the set $(\mathfrak{b} \times \mathfrak{b}) \cap \mathfrak{G}^\heartsuit$ is nonempty, for any Borel subalgebra \mathfrak{b} . Hence, this set is Zariski open and dense in $\mathfrak{b} \times \mathfrak{b}$. Furthermore, it follows from [CM], Lemma 7(i) and Lemma 8(i), that for any pair $(x, y) \in \mathfrak{G}^\heartsuit$ there is at most one Borel subalgebra that contains both x and y (in [CM], this result is attributed to Bolsinov [Bol]). Hence, the map μ restricts to a bijection $\mu^{-1}(\mathfrak{G}^\heartsuit) \xrightarrow{\sim} \mathfrak{G}^b \cap \mathfrak{G}^\heartsuit$. Both statements of the proposition follow from this since the map μ is proper. \square

6.2. A stratification of the isospectral commuting variety. Below, we will have to consider several reductive Lie algebras at the same time. To avoid confusion, we write $\mathfrak{C}(\mathfrak{l})$ for the commuting scheme of a reductive Lie algebra \mathfrak{l} and use similar notation for other objects associated with \mathfrak{l} .

Let $\mathfrak{N}(\mathfrak{l})$ be the *nilpotent* commuting variety of \mathfrak{l} , the variety of pairs of commuting *nilpotent* elements of \mathfrak{l} , equipped with reduced scheme structure. It is clear that we have $\mathfrak{N}(\mathfrak{l}) = \mathfrak{N}(\mathfrak{l}')$, where $\mathfrak{l}' := [\mathfrak{l}, \mathfrak{l}]$, the derived Lie algebra of \mathfrak{l} . According to [Pr], the irreducible components of $\mathfrak{N}(\mathfrak{l})$ are parametrized by the conjugacy classes of *distinguished* nilpotent elements of \mathfrak{l}' . The irreducible component corresponding to such a conjugacy class is equal to the closure in $\mathfrak{l}' \times \mathfrak{l}' = T^*(\mathfrak{l}')$ of the total space of the conormal bundle on that conjugacy class. It follows, in particular, that the dimension of each irreducible component of the variety $\mathfrak{N}(\mathfrak{l})$ equals $\dim \mathfrak{l}'$.

Fix a reductive connected group G and a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g} = \text{Lie } G$. Recall that the centralizer of an element of \mathfrak{t} is called a standard Levi subalgebra of \mathfrak{g} . Let S be the set of standard Levi subalgebras. Given a standard Levi subalgebra \mathfrak{l} , let $\mathfrak{t}_{\mathfrak{l}} \subset \mathfrak{t}$, resp. $\mathring{\mathfrak{T}}_{\mathfrak{l}} \subset \mathfrak{T}$, denote the set of elements $h \in \mathfrak{t}$, resp. $(h_1, h_2) \in \mathfrak{T}$, such that we have $\mathfrak{g}_h = \mathfrak{l}$, resp. $\mathfrak{g}_{h_1, h_2} = \mathfrak{l}$. Let $\mathfrak{t}_{\mathfrak{l}}$ denote the center of \mathfrak{l} . It is clear that $\mathfrak{t}_{\mathfrak{l}}$ is an irreducible Zariski open dense subset of $\mathfrak{t}_{\mathfrak{l}}$, resp. $\mathring{\mathfrak{T}}_{\mathfrak{l}}$ is an irreducible Zariski open dense subset of $\mathfrak{t}_{\mathfrak{l}} \times \mathfrak{t}_{\mathfrak{l}}$. We get a stratification $\mathfrak{t} = \sqcup_{\mathfrak{l} \in S} \mathfrak{t}_{\mathfrak{l}}$, resp. $\mathfrak{T} = \sqcup_{\mathfrak{l} \in S} \mathring{\mathfrak{T}}_{\mathfrak{l}}$.

Let $\mathfrak{X} = \mathfrak{X}(\mathfrak{g})$, the isospectral commuting variety of \mathfrak{g} , and let $p_{\mathfrak{x}} : \mathfrak{X}(\mathfrak{g}) \rightarrow \mathfrak{T}$ be the projection. For each standard Levi subalgebra \mathfrak{l} of \mathfrak{g} , we put $\mathfrak{X}_{\mathfrak{l}}(\mathfrak{g}) := (p_{\mathfrak{x}})^{-1}(\mathring{\mathfrak{T}}_{\mathfrak{l}})$. Thus, we get a partition $\mathfrak{X}(\mathfrak{g}) = \sqcup_{\mathfrak{l} \in S} \mathfrak{X}_{\mathfrak{l}}(\mathfrak{g})$ by G -stable locally closed, not necessarily smooth, subvarieties.

We consider a map $G \times \mathfrak{G} \times \mathfrak{T} \rightarrow \mathfrak{G} \times \mathfrak{T}$ given by the following assignment

$$g \times (y_1, y_2) \times (t_1, t_2) \mapsto (\text{Ad } g(y_1 + t_1), \text{Ad } g(y_2 + t_2)) \times (t_1, t_2). \quad (6.2.1)$$

Lemma 6.2.2. *For any standard Levi subalgebra \mathfrak{l} with Levi subgroup of $L \subset G$, the map (6.2.1) induces a G -equivariant isomorphism*

$$(G \times_L \mathfrak{N}(\mathfrak{l})) \times \mathring{\mathfrak{T}}_{\mathfrak{l}} \xrightarrow{\sim} \mathfrak{X}_{\mathfrak{l}}.$$

The second projection $(G \times_L \mathfrak{N}(\mathfrak{l})) \times \mathring{\mathfrak{T}}_{\mathfrak{l}} \rightarrow \mathring{\mathfrak{T}}_{\mathfrak{l}}$ goes, under the isomorphism, to the map $p_{\mathfrak{x}} : \mathfrak{X}_{\mathfrak{l}} \rightarrow \mathring{\mathfrak{T}}_{\mathfrak{l}}$. Thus, all irreducible components of the set $\mathfrak{X}_{\mathfrak{l}}$ have the same dimension equal to $\dim \mathfrak{g} + \dim \mathfrak{t}_{\mathfrak{l}}$

and are in one-to-one correspondence with the distinguished nilpotent conjugacy classes in the Lie algebra \mathfrak{l} .

Proof. Let $(x_1, x_2, t_1, t_2) \in \mathfrak{X}_{\mathfrak{l}}$. Lemma 6.1.1(ii) implies that, for any polynomial $f \in \mathbb{C}[\mathfrak{C}]^G$, we have $f(x_1, x_2) = f(h_{x_1}, h_{x_2})$. We know that the semisimple pair (h_{x_1}, h_{x_2}) is G -conjugate to an element of \mathfrak{T} , that W -invariant polynomials separate W -orbits in \mathfrak{T} , and that the restriction map (1.3.1) is surjective, [Jo]. It follows that the pair (h_{x_1}, h_{x_2}) is G -conjugate to the pair (t_1, t_2) . Hence, the pair (x_1, x_2) is G -conjugate to a pair of the form $(t_1 + y_1, t_2 + y_2)$ for some $(y_1, y_2) \in \mathfrak{N}(\mathfrak{l})$. The isomorphism of the lemma easily follows from this.

Now, for any irreducible component V of $\mathfrak{N}(\mathfrak{l})$, using the result of Premet mentioned above, we find

$$\begin{aligned} \dim((G \times_L V) \times \mathring{\mathfrak{X}}_{\mathfrak{l}}) &= \dim G - \dim L + \dim V + 2 \dim \mathfrak{t}_{\mathfrak{l}} \\ &= \dim \mathfrak{g} - \dim(\mathfrak{t}_{\mathfrak{l}} + [\mathfrak{l}, \mathfrak{l}]) + \dim[\mathfrak{l}, \mathfrak{l}] + 2 \dim \mathfrak{t}_{\mathfrak{l}} = \dim \mathfrak{g} + \dim \mathfrak{t}_{\mathfrak{l}}. \end{aligned}$$

This proves the dimension formula for irreducible components of the variety $\mathfrak{X}_{\mathfrak{l}}$. \square

We are now ready to complete the proof of Lemma 2.1.3.

Corollary 6.2.3. *The set \mathfrak{X}^{rs} is an irreducible and Zariski dense subset of \mathfrak{X} .*

Proof. Let \mathfrak{C}^{ss} be the set of semisimple pairs $(x, y) \in \mathfrak{C}$ and let $\mathfrak{X}_{\mathfrak{l}}^{ss} := \mathfrak{X}_{\mathfrak{l}} \cap p^{-1}(\mathfrak{C}^{ss})$. Lemma 6.2.2 implies that, for any Levi subalgebra \mathfrak{l} , any element $(x, y, t_1, t_2) \in \mathfrak{X}_{\mathfrak{l}}^{ss}$ is conjugate to the element (x', y', t_1, t_2) for some $(x', y') \in \mathring{\mathfrak{X}}_{\mathfrak{l}}$. In the special case where $\mathfrak{l} = \mathfrak{t}$, we have $\mathfrak{X}^{rs} = \mathfrak{X}_{\mathfrak{t}}$. Hence, Lemma 6.2.2 implies that \mathfrak{X}^{rs} is irreducible and, moreover, we have $\mathfrak{X}_{\mathfrak{l}}^{ss} \subset \overline{\mathfrak{X}^{rs}}$, for any standard Levi subalgebra \mathfrak{l} .

Thus, using the isomorphism of Lemma 6.2.2 one more time, we see that proving the Corollary reduces to showing that $\mathfrak{N}(\mathfrak{l})$, the nilpotent commuting variety of \mathfrak{l} , is contained in the closure of the set $\mathfrak{C}^{ss}(\mathfrak{l})$. But we have $\mathfrak{C}^{ss}(\mathfrak{l}) \supseteq \mathfrak{C}^{rs}(\mathfrak{l})$ and the set $\mathfrak{C}^{rs}(\mathfrak{l})$ is dense in $\mathfrak{C}(\mathfrak{l})$, by Proposition 2.1.1(i); explicitly, this is Corollary 4.7 from [Ri1]. The result follows. \square

6.3. Proof of Lemma 2.6.4. We have the projection $p : \mathfrak{X}(\mathfrak{g}) \rightarrow \mathfrak{C}(\mathfrak{g})$ and, for any standard Levi subalgebra $\mathfrak{l} \subset \mathfrak{g}$, put $\mathfrak{C}_{\mathfrak{l}}(\mathfrak{g}) = p(\mathfrak{X}_{\mathfrak{l}}(\mathfrak{g}))$ where we use the notation of Lemma 6.2.2. Thus, one has $\mathfrak{C}(\mathfrak{g}) = \bigcup_{\mathfrak{l} \in \mathcal{S}} \mathfrak{C}_{\mathfrak{l}}(\mathfrak{g})$. Note that, for a pair of standard Levi subalgebras $\mathfrak{l}_1, \mathfrak{l}_2 \subset \mathfrak{g}$, the corresponding pieces $\mathfrak{C}_{\mathfrak{l}_1}(\mathfrak{g})$ and $\mathfrak{C}_{\mathfrak{l}_2}(\mathfrak{g})$ are equal whenever the Levi subalgebras \mathfrak{l}_1 and \mathfrak{l}_2 are conjugate in \mathfrak{g} ; otherwise these two pieces are disjoint.

Let \mathfrak{l} be a standard Levi subalgebra in \mathfrak{g} . We are interested in the dimension of the set $\mathfrak{C}_{\mathfrak{l}}(\mathfrak{g}) \setminus \mathfrak{C}^{rr}$. The dimension formula of Lemma 6.2.2 shows that the codimension of the set $\mathfrak{C}_{\mathfrak{l}}(\mathfrak{g})$ in $\mathfrak{C}(\mathfrak{g})$ equals $\dim(\mathfrak{t}/\mathfrak{t}_{\mathfrak{l}})$. We see that to prove Lemma 2.6.4 it suffices to show that, for any standard Levi subalgebra $\mathfrak{l} \subset \mathfrak{g}$ such that the codimension of $\mathfrak{t}_{\mathfrak{l}}$ in \mathfrak{t} equals either 0 or 1, the set $\mathfrak{C}_{\mathfrak{l}}(\mathfrak{g}) \setminus \mathfrak{C}^{rr}$ has codimension ≥ 2 in $\mathfrak{C}(\mathfrak{g})$.

In the first case we have $\mathfrak{t} = \mathfrak{t}_{\mathfrak{l}} = \mathfrak{l}$. Thus, one has $\mathfrak{N}(\mathfrak{l}) = \{0\}$, so $\mathfrak{N}(\mathfrak{l}) + \mathring{\mathfrak{X}}_{\mathfrak{l}} = \mathring{\mathfrak{X}}$, where we have used simplified notation $\mathring{\mathfrak{X}} := \mathring{\mathfrak{X}}_{\mathfrak{t}}$. The set $\mathring{\mathfrak{X}} \setminus \mathfrak{C}^{rr}$ consists of the pairs $(h_1, h_2) \in \mathring{\mathfrak{X}}$ such that neither h_1 nor h_2 is regular. Therefore, each of these two elements belongs to some root hyperplane in \mathfrak{t} , that is, belongs to a finite union of codimension 1 subspaces in \mathfrak{t} . We conclude that the set $\mathring{\mathfrak{X}} \setminus \mathfrak{C}^{rr}$ has codimension ≥ 2 in $\mathfrak{N}(\mathfrak{l}) + \mathring{\mathfrak{X}}_{\mathfrak{l}}$, as required.

Next, let $\dim(\mathfrak{t}/\mathfrak{t}_{\mathfrak{l}}) = 1$. In that case, \mathfrak{l} is a *minimal* Levi subalgebra of \mathfrak{g} . Thus, there is a root $\alpha \in \mathfrak{t}^*$ in the root system of $(\mathfrak{g}, \mathfrak{t})$ such that $\mathfrak{t}_{\mathfrak{l}} = \text{Ker } \alpha$ is a codimension 1 hyperplane in \mathfrak{t} . We have $\mathfrak{l} = \mathfrak{t}_{\mathfrak{l}} \oplus [\mathfrak{l}, \mathfrak{l}]$ where $[\mathfrak{l}, \mathfrak{l}]$ is an \mathfrak{sl}_2 -subalgebra of \mathfrak{g} associated with the root α . It is easy to see that $\mathfrak{N}(\mathfrak{sl}_2)$ is an irreducible variety formed by the pairs of nilpotent elements

proportional to each other (the zero element is declared to be proportional to any element). Thus, $\mathfrak{N}(\mathfrak{l}) + \mathring{\mathfrak{Z}}_{\mathfrak{l}}$ is an irreducible variety.

To complete the proof of the lemma in this case, we must show that the complement of the set $U = (\mathfrak{N}(\mathfrak{l}) + \mathring{\mathfrak{Z}}_{\mathfrak{l}}) \cap \mathfrak{C}^{rr}$ has codimension ≥ 1 in $\mathfrak{N}(\mathfrak{l}) + \mathring{\mathfrak{Z}}_{\mathfrak{l}}$. The set U is a Zariski open subset in an irreducible variety. Thus, it suffices to show that the set U is nonempty. For this, pick $h \in \mathring{\mathfrak{l}}_1$ and let $n \in \mathfrak{sl}_2$ be any nonzero nilpotent element. Then, n is a regular element of the Lie algebra $[\mathfrak{l}, \mathfrak{l}] = \mathfrak{sl}_2$; hence $h + n$ is a regular element of \mathfrak{g} . Therefore, we have $(h + n, h + n) \in U$, and we are done. \square

6.4. Proof of Theorem 1.5.2(i). The property of a coherent sheaf be Cohen-Macaulay is stable under taking direct images by finite morphisms and taking direct summands. Thus, Theorem 1.3.4 implies that the sheaf \mathcal{R} , as well as the isotypic components \mathcal{R}^E for any W -module E , is Cohen-Macaulay.

The scheme $\mathfrak{X}_{\text{norm}}/W$ is reduced and integrally closed, as a quotient of an integrally closed reduced scheme by a finite group action, [Kr], §3.3. Further, by Lemma 2.1.3(ii), the map p_{norm} is generically a Galois covering with W being the Galois group. It follows that the induced map $\mathfrak{X}_{\text{norm}}/W \rightarrow \mathfrak{C}_{\text{norm}}$ is a finite and birational morphism of normal varieties. Hence it is an isomorphism, that is, the canonical morphism $\mathcal{O}_{\mathfrak{C}_{\text{norm}}} \rightarrow ((p_{\text{norm}})_* \mathcal{O}_{\mathfrak{X}_{\text{norm}}})^W = \mathcal{R}^W$ is an isomorphism. This yields an isomorphism $\mathfrak{X}_{\text{norm}}/W \cong \mathfrak{C}_{\text{norm}}$ and implies Corollary 1.3.5.

Observe next that the sheaf $\mathcal{R}|_{\mathfrak{C}^r}$ is a Cohen-Macaulay sheaf on a smooth variety, hence it is locally free. The fiber of the corresponding algebraic vector bundle over any point of the open set $\mathfrak{C}^{rs} \subset \mathfrak{C}^r$ affords the regular representation of the Weyl group W , by Lemma 2.1.3(ii). The statement of Theorem 1.5.2(i) follows from this by continuity since the set \mathfrak{C}^{rs} is dense in \mathfrak{C}^r .

6.5. Proof of Theorem 1.5.2(ii) and of Theorem 1.6.1(i). We use the notation of Definition 2.6.3 and let $\tilde{\mathfrak{X}}_1 := \mu^{-1}(\mathfrak{C}_1)$. Write $\tilde{q} : \tilde{\mathfrak{X}}_1 \rightarrow \tilde{\mathfrak{g}}^r$, $(\mathfrak{b}, x, y) \mapsto (\mathfrak{b}, x)$, resp. $q_1 : \mathfrak{C}_1 \rightarrow \mathfrak{g}^r$, $(x, y) \mapsto x$, for the natural projection, and let π be the map from Corollary 3.3.5. Thus, one has a commutative diagram

$$\begin{array}{ccccc} \tilde{\mathfrak{X}}_1 & \xrightarrow[\sim]{\pi} & \mathfrak{X}_1 & \xrightarrow{p} & \mathfrak{C}_1 \\ \downarrow \tilde{q} & & \downarrow q & & \downarrow q_1 \\ \tilde{\mathfrak{g}}^r & \xrightarrow[\sim]{\pi} & \mathfrak{r} & \xrightarrow{pr} & \mathfrak{g}^r \end{array} \quad (6.5.1)$$

Lemma 6.5.2. *The right square in diagram (6.5.1) is cartesian.*

Proof. It is immediate from Lemma 5.4.1 that the map $\tilde{q} \times (p \circ \pi) : \tilde{\mathfrak{X}}_1 \rightarrow \tilde{\mathfrak{g}}^r \times_{\mathfrak{g}^r} \mathfrak{C}_1$ that results from the diagram is a set theoretic bijection. Further, all the varieties involved in diagram (6.5.1) are smooth and the map q_1 is a smooth morphism (the vector bundle projection $N_{\mathfrak{r}} \rightarrow \mathfrak{r}$, cf. Lemma 2.1.5). It follows that $\tilde{\mathfrak{g}}^r \times_{\mathfrak{g}^r} \mathfrak{C}_1$ is a smooth variety and, moreover, the bijection above gives an isomorphism $\tilde{\mathfrak{X}}_1 \xrightarrow{\sim} \tilde{\mathfrak{g}}^r \times_{\mathfrak{g}^r} \mathfrak{C}_1$, of algebraic varieties. We deduce that the rectangle along the perimeter of diagram (6.5.1) is a cartesian square.

To complete the proof, we observe that the map π in (6.5.1) is an isomorphism by Corollary 3.3.5, resp. the map π is an isomorphism by Proposition 3.1.1(i). We conclude that the right square in diagram (6.5.1) is cartesian as well. \square

The morphism pr in diagram (6.5.1) is finite and the morphism q_1 is smooth. So, thanks to Lemma 6.5.2, we may apply smooth base change for the cartesian square on the right of (6.5.1). Combining this with smooth base change for the cartesian square on the right of diagram (5.3.1), yields a chain of natural $G \times W \times \mathbb{C}^\times \times \mathbb{C}^\times$ -equivariant sheaf isomorphisms

$$\mathcal{R}|_{\mathfrak{C}_1} = p_* \mathcal{O}_{\mathfrak{X}_1} = p_* q^* \mathcal{O}_t = q_1^* pr_* \mathcal{O}_t = q_1^* pr_*(\tilde{\gamma}^* \mathcal{O}_t) = q_1^* \gamma^*(\vartheta_* \mathcal{O}_t). \quad (6.5.3)$$

Therefore, for any W -representation E , we deduce

$$\mathcal{R}^E|_{\mathfrak{C}_1} = (E \otimes q_1^* pr_* \mathcal{O}_t)^W = q_1^*((E \otimes pr_* \mathcal{O}_t)^W) = q_1^* \gamma^*((E \otimes \vartheta_* \mathcal{O}_t)^W). \quad (6.5.4)$$

Observe next that, for any $(x, y) \in \mathfrak{C}_1$, the Lie algebra \mathfrak{g}_x is abelian, hence we have $\mathfrak{g}_{x,y} = \mathfrak{g}_x \cap \mathfrak{g}_y = \mathfrak{g}_x$. From this, writing $\mathbf{g}_1 := \mathbf{g}_{\mathfrak{C}_1}$ for short, we get a natural isomorphism $\mathbf{g}_1 = q_1^* \mathbf{g}_{\mathfrak{g}^r}$, of sheaves on \mathfrak{C}_1 . We deduce that the sheaf \mathbf{g}_1 is a locally free. Furthermore, applying to functor $q_1^*(-)$ to the isomorphism of Theorem 5.4.6(i) and using (6.5.4) one obtains the following isomorphisms

$$\wedge^k \mathbf{g}_1 = q_1^*(\wedge^k \mathbf{g}_{\mathfrak{g}^r}) \xrightarrow{\sim} q_1^*((\wedge^k \mathfrak{t} \otimes pr_* \mathcal{O}_t)^W) = \mathcal{R}^{\wedge^k \mathfrak{t}}|_{\mathfrak{C}_1}. \quad (6.5.5)$$

Now, let L be a rational G -module such that the set of weights of L is contained in the root lattice. Then, a similar argument yields a natural isomorphism $L^{\mathbf{g}_1} = q_1^*(L^{\mathbf{g}_{\mathfrak{g}^r}})$. Therefore the sheaf $L^{\mathbf{g}_1}$ is locally free. Moreover, applying the functor $q_1^*(-)$ to the isomorphism of Theorem 5.4.6(ii) and using (6.5.4) again, one similarly obtains an isomorphism

$$L^{\mathbf{g}_1} \xrightarrow{\sim} \mathcal{R}^{L^{\mathfrak{h}}}|_{\mathfrak{C}_1}. \quad (6.5.6)$$

Similar considerations apply, of course, in the case where the set \mathfrak{C}_1 is replaced by the set \mathfrak{C}_2 . It follows, in particular, that $\mathbf{g}_{rr} := \mathbf{g}_{\mathfrak{C}^{rr}}$ and $L^{\mathbf{g}_{rr}} := L^{\mathbf{g}_{\mathfrak{C}^{rr}}}$ are locally free coherent sheaves on $\mathfrak{C}^{rr} = \mathfrak{C}_1 \cup \mathfrak{C}_2$. However, it is not clear *a priori*, that the ‘ \mathfrak{C}_2 -counterparts’ of morphisms (6.5.5)-(6.5.6) agree with those in (6.5.5)-(6.5.6) on the overlap $\mathfrak{C}_1 \cap \mathfrak{C}_2$.

To overcome this difficulty, we now produce an independent direct construction of canonical morphisms

$$\lambda_{rr}^k : \wedge^k \mathbf{g}_{\mathfrak{C}^{rr}} \rightarrow \mathcal{R}^{\wedge^k \mathfrak{t}}|_{\mathfrak{C}^{rr}}, \quad \text{resp.} \quad \lambda_{rr}^L : L^{\mathbf{g}_{rr}} \rightarrow \mathcal{R}^{L^{\mathfrak{h}}}|_{\mathfrak{C}^{rr}}. \quad (6.5.7)$$

This will be done by adapting the strategy of §5.3 as follows. Let $\mathfrak{X}^{rr} = p^{-1}(\mathfrak{C}^{rr})$ and $\tilde{\mathfrak{X}}^{rr} = \mu^{-1}(\mathfrak{C}^{rr})$. We know by the above that the sheaf $\mu^* \mathbf{g}_{rr}$, resp. $\mu^*(L^{\mathbf{g}_{rr}})$, is locally free, being a pull-back of a locally free coherent sheaf on \mathfrak{C}^{rr} .

Now let $(x, y, \mathfrak{b}) \in \tilde{\mathfrak{X}}^{rr}$. Then, $x, y \in \mathfrak{b}$ and, moreover, we have that either $\mathfrak{g}_{x,y} = \mathfrak{g}_x$ or $\mathfrak{g}_{x,y} = \mathfrak{g}_y$. In each of the two cases, applying Lemma 5.4.1, we deduce an inclusion $L^{\mathfrak{g}_{x,y}} \subset L^{\mathfrak{g}_0}$. Therefore, one gets, as in §5.3, a well defined morphism $\mu^*(L^{\mathbf{g}_{rr}}) \rightarrow L^{\mathfrak{h}} \otimes \mathcal{O}_{\tilde{\mathfrak{X}}^{rr}}$. We may further transport this morphism via π , the isomorphism of Corollary 3.3.5(ii). This way, one constructs a canonical morphism $f : p^*(L^{\mathbf{g}_{rr}}) \rightarrow L^{\mathfrak{h}} \otimes \mathcal{O}_{\mathfrak{X}^{rr}}$. The morphisms in (6.5.7) are now defined from the morphism f , by adjunction, mimicing the construction of §5.3.

Lemma 6.5.8. *The restriction of the morphism λ_{rr}^k , resp. λ_{rr}^L , in (6.5.7), to the open set $\mathfrak{C}_1 \subset \mathfrak{C}^{rr}$ reduces to isomorphism (6.5.5), resp. (6.5.6).*

Similar claim holds in the case of the set $\mathfrak{C}_2 \subset \mathfrak{C}^{rr}$.

Proof. All the sheaves involved are G -equivariant and locally free. Hence, it suffices to check the statement of the lemma fiberwise, and only at the points of the form $(x, y) \in \mathfrak{t}^r \times \mathfrak{t}^r$. In that case verification is straightforward and is left for the reader. \square

Lemma 6.5.8 implies that each of the morphisms in (6.5.7) is an isomorphism of locally free sheaves on \mathfrak{C}^{rr} .

To complete the proof of Theorem 1.6.1(i) write $j : \mathfrak{C}^{rr} \hookrightarrow \mathfrak{C}_{\text{norm}}$ be the open imbedding. We know that the sheaf \mathcal{R}^E is Cohen-Macaulay for any W -representation E . It follows, that the canonical morphism $\mathcal{R}^E \rightarrow j_*(\mathcal{R}^E|_{\mathfrak{C}^r})$ is an isomorphism, cf. Lemma 2.6.1. Similarly, by Lemma 5.1.2(ii), we have a canonical isomorphism $L^{\mathfrak{g}_{\mathfrak{C}_{\text{norm}}}} \xrightarrow{\sim} j_*(L^{\mathfrak{g}_{\mathfrak{C}^r}})$. Thus, applying the functor $j_*(-)$ to the isomorphism $L^{\mathfrak{g}_{\mathfrak{C}^r}} \cong \mathcal{R}^{L^{\mathfrak{h}}}|_{\mathfrak{C}^r}$ proved earlier, we deduce $L^{\mathfrak{g}_{\mathfrak{C}_{\text{norm}}}} = j_*(L^{\mathfrak{g}_{\mathfrak{C}^r}}) \cong j_*(\mathcal{R}^{L^{\mathfrak{h}}}|_{\mathfrak{C}^r}) \cong \mathcal{R}^{L^{\mathfrak{h}}}$ and Theorem 1.6.1(i) follows.

A similar (even simpler) argument proves Theorem 1.5.2(ii).

Remark 6.5.9. In the case of a not necessarily small representation L , one can still obtain a description of the image of the corresponding morphism λ_{rr}^L , in (6.5.7), in the spirit of Theorem 5.7.1. That description is not very useful, however, since the sheaf $L^{\mathfrak{g}_{\mathfrak{C}}}|_{\mathfrak{C}^r}$ turns out to be *not* locally free, in general, already for $\mathfrak{g} = \mathfrak{sl}_2$.

6.6. Proof of Corollary 1.5.1. The isomorphism $\mathcal{O}_{\mathfrak{C}_{\text{norm}}} \cong \mathcal{R}^W$, in Corollary 1.5.1(i), has been already established at the beginning of §6.4.

Proof of the isomorphism $\mathcal{K}_{\mathfrak{C}^r} \cong \mathcal{R}^{\text{sign}}|_{\mathfrak{C}^r}$. Given a point $(x, y) \in \mathfrak{C}$, let $T_{x,y}\mathfrak{C}$, resp. $T_{x,y}^*\mathfrak{C}$, be the Zariski tangent, resp. cotangent, space to \mathfrak{C} at (x, y) . Let $\kappa_* : \mathfrak{g} \oplus \mathfrak{g} \rightarrow \mathfrak{g}$ be the differential of the commutator map κ at the point (x, y) . Then, one has an exact sequence of vector spaces

$$0 \longrightarrow T_{x,y}\mathfrak{C} \longrightarrow \mathfrak{g} \oplus \mathfrak{g} \xrightarrow{\kappa_*} \mathfrak{g} \longrightarrow \text{Coker}(\kappa_*) \longrightarrow 0.$$

We use an invariant form on \mathfrak{g} to identify \mathfrak{g}^* with \mathfrak{g} and write κ_*^\top for the linear map dual to the map κ_* . Then, dualizing the exact sequence above yields an exact sequence

$$0 \longleftarrow T_{x,y}^*\mathfrak{C} \longleftarrow \mathfrak{g} \oplus \mathfrak{g} \xleftarrow{\kappa_*^\top} \mathfrak{g} \longleftarrow \text{Ker}(\kappa_*^\top) \longleftarrow 0 \quad (6.6.1)$$

Now, the map κ_* is given by the formula $\kappa_* : (u, v) \mapsto [x, u] - [y, v]$. Using the invariance of the bilinear form one easily finds that the dual map is given by the formula $\kappa_*^\top : a \mapsto [x, a] \oplus [y, a]$. We conclude that $\text{Ker}(\kappa_*^\top) = \mathfrak{g}_{x,y}$.

Write \det for the top exterior power of a vector space. From (6.6.1), we deduce a canonical isomorphism $\det T_{x,y}^*\mathfrak{C} = \det(\mathfrak{g} \oplus \mathfrak{g}) \otimes (\det \mathfrak{g})^{-1} \otimes \det \mathfrak{g}_{x,y}$. For $(x, y) \in \mathfrak{C}^r$, we have $\det \mathfrak{g}_{x,y} = \wedge^{\mathfrak{r}} \mathfrak{g}_{x,y} = \wedge^{\mathfrak{r}} \mathfrak{g}_{\mathfrak{C}}|_{(x,y)}$. Therefore, a choice of base vector in the 1-dimensional vector space $\det \mathfrak{g}$ determines, for all $(x, y) \in \mathfrak{C}^r$, an isomorphism $\det T_{x,y}^*\mathfrak{C} \cong \wedge^{\mathfrak{r}} \mathfrak{g}_{\mathfrak{C}}|_{(x,y)}$. This yields an isomorphism $\mathcal{K}_{\mathfrak{C}^r} \cong \wedge^{\mathfrak{r}} \mathfrak{g}_{\mathfrak{C}^r}$ of locally free sheaves. \square

We can now complete the proof of Corollary 1.5.1. We know that $\mathfrak{C}_{\text{norm}}$ is a Cohen-Macaulay variety and that the sheaf \mathcal{R} on $\mathfrak{C}_{\text{norm}}$ is isomorphic to its Grothendieck dual $R\mathcal{H}om_{\mathcal{O}_{\mathfrak{C}_{\text{norm}}}}(\mathcal{R}, \mathcal{K}_{\mathfrak{C}_{\text{norm}}})$. It follows that for any $j \neq 0$ one has $R^j\mathcal{H}om_{\mathcal{O}_{\mathfrak{C}_{\text{norm}}}}(\mathcal{R}, \mathcal{K}_{\mathfrak{C}_{\text{norm}}}) = 0$ and, moreover, there is an isomorphism $\mathcal{R} \cong \mathcal{H}om_{\mathcal{O}_{\mathfrak{C}_{\text{norm}}}}(\mathcal{R}, \mathcal{K}_{\mathfrak{C}_{\text{norm}}})$. Further, since $\mathcal{O}_{\mathfrak{C}_{\text{norm}}}$ is a direct summand of \mathcal{R} the sheaf $\mathcal{K}_{\mathfrak{C}_{\text{norm}}}$ is a direct summand of $\mathcal{H}om_{\mathcal{O}_{\mathfrak{C}_{\text{norm}}}}(\mathcal{R}, \mathcal{K}_{\mathfrak{C}_{\text{norm}}}) \cong \mathcal{R}$. Hence, $\mathcal{K}_{\mathfrak{C}_{\text{norm}}}$ is a Cohen-Macaulay sheaf. Similarly, the sheaf $\mathcal{R}^{\text{sign}}$ is also Cohen-Macaulay.

Recall that two Cohen-Macaulay sheaves are isomorphic if and only if they have isomorphic restrictions to a complement of a closed subset of codimension ≥ 2 . The isomorphism $\mathcal{K}_{\mathfrak{C}_{\text{norm}}} \cong \mathcal{R}^{\text{sign}}$ of Corollary 1.5.1(i) now follows from the chain of isomorphisms $\mathcal{K}_{\mathfrak{C}_{\text{norm}}}|_{\mathfrak{C}^r} \cong \wedge^{\mathfrak{r}} \mathfrak{g}_{\mathfrak{C}^r} \cong (\wedge^{\mathfrak{r}} \mathfrak{t} \otimes \mathcal{R})^W|_{\mathfrak{C}^r} = \mathcal{R}^{\text{sign}}|_{\mathfrak{C}^r}$ where the second isomorphism holds by Theorem 1.5.2(ii).

Finally, the isomorphism of part (ii) of Corollary 1.5.1 follows, thanks to the isomorphism $\mathcal{K}_{\mathfrak{C}_{\text{norm}}} \cong \mathcal{R}^{\text{sign}}$, by equating the corresponding W -isotypic components on each side of the self-duality isomorphism $\text{Hom}_{\mathcal{O}_{\mathfrak{C}_{\text{norm}}}}(\mathcal{R}, \mathcal{R}^{\text{sign}}) \cong \mathcal{R}$ proved earlier. \square

Remark 6.6.2. The short exact sequence (6.6.1) implies that, for any $(x, y) \in \mathfrak{C}$, one has

$$\mathbf{r} + \dim \mathfrak{g} = \dim \mathfrak{C} \leq \dim T_{x,y}^* \mathfrak{C} = 2 \dim \mathfrak{g} - (\dim \mathfrak{g} - \dim \text{Ker } \kappa_*^\top) = \dim \mathfrak{g} + \dim \mathfrak{g}_{x,y},$$

where in the first equality we have used Proposition 2.1.1(i). We deduce an inequality $\mathbf{r} \leq \dim \mathfrak{g}_{x,y}$. Moreover, we see that this inequality becomes an equality if and only if (x, y) is a smooth point of the scheme \mathfrak{C} . This proves Proposition 2.1.1(ii).

6.7. Proof of Theorem 1.6.1(ii)-(iii). We have the following chain of natural W -equivariant algebra maps

$$\begin{aligned} \mathbb{C}[\mathfrak{T}] &= \mathbb{C}[\mathfrak{T}]^W \otimes_{\mathbb{C}[\mathfrak{T}]^W} \mathbb{C}[\mathfrak{T}] = \mathbb{C}[\mathfrak{C}_{\text{red}}]^G \otimes_{\mathbb{C}[\mathfrak{C}_{\text{red}}]^G} \mathbb{C}[\mathfrak{T}] \\ &= \mathbb{C}[\mathfrak{C}_{\text{red}} \times_{\mathfrak{C}_{\text{red}}//G} \mathfrak{T}]^G \twoheadrightarrow \mathbb{C}[\mathfrak{X}]^G \hookrightarrow \mathbb{C}[\mathfrak{X}_{\text{norm}}]^G. \end{aligned}$$

Let f be the composition of the above maps. It follows from the isomorphism $\mathfrak{C}^{rs} \cong G \times_{N(T)} \mathring{\mathfrak{T}}_{\mathfrak{t}}$ that the map f induces an isomorphism between the fields of fractions of the algebras $\mathbb{C}[\mathfrak{T}]$ and $\mathbb{C}[\mathfrak{X}_{\text{norm}}]^G$, respectively. The algebra $\mathbb{C}[\mathfrak{X}_{\text{norm}}]^G$ is, by definition, a finitely generated $\mathbb{C}[\mathfrak{X}]^G$ -module, hence, also a finitely generated module over the image of f .

Thus, since the algebra $\mathbb{C}[\mathfrak{T}]$ is integrally closed, we obtain $\Gamma(\mathfrak{C}_{\text{norm}}, \mathcal{R})^G = \mathbb{C}[\mathfrak{X}_{\text{norm}}]^G = \mathbb{C}[\mathfrak{T}]$. Equating W -isotypic components on each side of this isomorphism yields the first isomorphism of Theorem 1.6.1(ii).

To prove the second isomorphism, we compute

$$\begin{aligned} (L \otimes \mathbb{C}[\mathfrak{C}_{\text{norm}}])^G &= \Gamma(\mathfrak{C}_{\text{norm}}, L \otimes \mathcal{O}_{\mathfrak{C}_{\text{norm}}})^G \\ &= \Gamma(\mathfrak{C}_{\text{norm}}, L^{\mathfrak{g}_{\mathfrak{C}_{\text{norm}}}})^G \quad \text{by Lemma 5.1.2(i)} \\ &= \Gamma(\mathfrak{C}_{\text{norm}}, \mathcal{R}^{L^{\mathfrak{h}}})^G \quad \text{by Theorem 1.6.1(i)} \\ &= (L^{\mathfrak{h}} \otimes \mathbb{C}[\mathfrak{T}])^W \quad \text{by the previous paragraph.} \end{aligned}$$

It is immediate to check, by restricting to the open set of regular semisimple pairs, that the composition of the chain of isomorphism above goes, via the identification $L^{\mathfrak{h}} = L^{\mathfrak{t}}$ induced by the imbedding $i : \mathfrak{t} \hookrightarrow \mathfrak{g}$, to the restriction homomorphism $i^* : (L \otimes \mathbb{C}[\mathfrak{C}_{\text{norm}}])^G \rightarrow (L^{\mathfrak{t}} \otimes \mathbb{C}[\mathfrak{T}])^W$. This proves part (ii) of Theorem 1.6.1.

To prove part (iii) we use Theorem 1.5.2(ii). From that theorem, we deduce $\Gamma(\mathfrak{C}^r, \mathcal{R}^{\wedge^{st}}) \cong \Gamma(\mathfrak{C}^r, \wedge^s \mathfrak{g}_{\mathfrak{C}^r})$, for any $s \geq 0$. Further, we know that the set $\mathfrak{C}_{\text{norm}} \setminus \mathfrak{C}^r$ has codimension ≥ 2 in $\mathfrak{C}_{\text{norm}}$ and that $\mathcal{R}^{\wedge^{st}}$ is a Cohen-Macaulay sheaf on $\mathfrak{C}_{\text{norm}}$. It follows that the natural restriction map induces an isomorphism $\Gamma(\mathfrak{C}_{\text{norm}}, \mathcal{R}^{\wedge^{st}}) \xrightarrow{\sim} \Gamma(\mathfrak{C}^r, \mathcal{R}^{\wedge^{st}})$. The proof is now completed by the following chain of isomorphisms:

$$\Gamma(\mathfrak{C}^r, \wedge^s \mathfrak{g}_{\mathfrak{C}^r})^G \cong \Gamma(\mathfrak{C}^r, \mathcal{R}^{\wedge^{st}})^G \cong \Gamma(\mathfrak{C}_{\text{norm}}, \mathcal{R}^{\wedge^{st}})^G \cong (\wedge^s \mathfrak{t} \otimes \mathbb{C}[\mathfrak{T}])^W. \quad \square$$

7. PRINCIPAL NILPOTENT PAIRS

7.1. Filtrations and Rees modules. Given a vector space E we refer to a direct sum decomposition $E = \bigoplus_{i,j \geq 0} E^{i,j}$ as a *bigrading* on E . Similarly, a collection of subspaces

$F_{i,j}E \subset E$, $i, j \geq 0$ such that $F_{i,j}E \subset F_{i',j'}E$ whenever $i \leq i'$ and $j \leq j'$ will be referred to as a *bifiltration* on E . Canonically associated with a bifiltration $F_{i,j}E$, there is a pair of bigraded vector spaces

$$\text{gr } E := \bigoplus_{i,j \geq 0} \text{gr}_{i,j} E, \quad \text{gr}_{i,j} E := \frac{F_{i,j}E}{F_{i-1,j}E + F_{i,j-1}E}, \quad \text{resp.} \quad RE := \bigoplus_{i,j \geq 0} F_{i,j}E. \quad (7.1.1)$$

We view the polynomial algebra $\mathbb{C}[\tau_1, \tau_2]$ as bigraded algebra such that $\deg \tau_1 = (1, 0)$ and $\deg \tau_2 = (0, 1)$. Below, we will often make no distinction between RE and the *Rees module* of E defined as $\sum_{i,j} \tau_1^i \tau_2^j \cdot F_{i,j}E \subset \mathbb{C}[\tau_1, \tau_2] \otimes E$, a bigraded $\mathbb{C}[\tau_1, \tau_2]$ -submodule of a free $\mathbb{C}[\tau_1, \tau_2]$ -module with generators E . There is a canonical bigraded space isomorphism

$$\text{gr } E \cong RE / (\tau_1 \cdot RE + \tau_2 \cdot RE). \quad (7.1.2)$$

Given a vector space E equipped with a pair of ascending filtrations $'F.E$ and $''F.E$, there are Rees modules $'RE := \sum_i \tau_1^i \cdot 'F_i E \subset \mathbb{C}[\tau_1] \otimes E$, resp. $''RE := \sum_j \tau_2^j \cdot ''F_j E \subset \mathbb{C}[\tau_2] \otimes E$, as well as associated graded spaces $'\text{gr } E$, resp. $''\text{gr } E$. One may further define a bifiltration on E by the formula $F_{i,j}E := 'F_i E \cap ''F_j E$, $i, j \geq 0$. One has the corresponding Rees $\mathbb{C}[\tau_1, \tau_2]$ -module RE . There are canonical isomorphisms

$$\mathbb{C}[\tau_2^{\pm 1}] \otimes_{\mathbb{C}[\tau_2]} RE \cong \mathbb{C}[\tau_2^{\pm 1}] \otimes 'RE, \quad \text{resp.} \quad \text{gr } E \cong '\text{gr}(''\text{gr } E) \cong ''\text{gr}(''\text{gr } E),$$

of graded $\mathbb{C}[\tau_1, \tau_2^{\pm 1}]$ -modules, resp. bigraded vector spaces. Specializing the first of the above isomorphisms at the point $(\tau_1, \tau_2) = (0, 1)$ and using that $'RE/\tau_1 \cdot 'RE \cong '\text{gr } E$ we deduce a canonical isomorphism of graded vector spaces

$$RE / (\tau_1 \cdot RE + (\tau_2 - 1) \cdot RE) \cong '\text{gr } E. \quad (7.1.3)$$

Now let $E = \bigoplus_{i,j} E^{i,j}$ be a bigraded vector space itself. Associated naturally with the bigrading, there are two filtrations on E defined by $'F_m E := \bigoplus_{\{i,j \mid i \leq m\}} E^{i,j}$ and $''F_n E := \bigoplus_{\{i,j \mid j \leq n\}} E^{i,j}$, respectively. Let $F_{m,n} E := 'F_m E \cap ''F_n E = \bigoplus_{\{i,j \mid i \leq m, j \leq n\}} E^{i,j}$ be the corresponding bifiltration. Further, equip the $\mathbb{C}[\tau_1, \tau_2]$ -module $\mathbb{C}[\tau_1, \tau_2] \otimes E$ with a standard tensor product bigrading $(\mathbb{C}[\tau_1, \tau_2] \otimes E)^{p,q} := \sum_{\{0 \leq i \leq p, 0 \leq j \leq q\}} \mathbb{C}^{i,j}[\tau_1, \tau_2] \otimes E^{p-i, q-j}$.

Lemma 7.1.4. *For a bigraded vector space $E = \bigoplus_{i,j} E^{i,j}$, the assignment*

$$\aleph : \tau_1^m \tau_2^n \otimes u_{i,j} \longmapsto \tau_1^{m+i} \tau_2^{n+j} \cdot u_{i,j}, \quad u_{i,j} \in E^{i,j}, \quad i, j, m, n \geq 0,$$

yields a bigraded $\mathbb{C}[\tau_1, \tau_2]$ -module isomorphism $\aleph : \mathbb{C}[\tau_1, \tau_2] \otimes E \xrightarrow{\sim} RE$. \square

This lemma is clear. Later on, we will use the following simple result

Corollary 7.1.5. *Let E be a finite dimensional vector space equipped with a pair of ascending filtrations $'F.E$ and $''F.E$, respectively. Then RE is a finite rank free $\mathbb{C}[\tau_1, \tau_2]$ -module.*

Proof. The filtrations $'F.E$ and $''F.E$ form a pair of partial flags in E . Hence, applying the Bruhat lemma for pairs of flags, we deduce that there exists a bigrading $E = \bigoplus_{m,n} E^{m,n}$ such that the original filtrations are associated, as has been explained before Lemma 7.1.4, with that bigrading. The corollary now follows from the lemma. \square

Remark 7.1.6. For a general bifiltration on a finite dimensional vector space E that does not come from a pair of filtrations one may have $\dim(\text{gr } E) > \dim E$, so the $\mathbb{C}[\tau_1, \tau_2]$ -module RE is not necessarily flat, in general. \diamond

Next, let $I \subset E$ be a subspace of a vector subspace E and put $A = E/I$. Write $\overline{}$ for the image of a vector subspace $V \subset E$ under the projection $E \rightarrow E/I$. A filtration on E induces a quotient filtration on A . Therefore, given a pair of filtrations $'F \cdot E$ and $''F \cdot E$ on E one has the corresponding quotient filtrations $F \cdot A = ('F \cdot E + I)/I$ and $''F \cdot A = (''F \cdot E + I)/I$ on A .

There are two, *potentially different*, ways to define a bifiltration on A as follows

$$\begin{aligned} F_{i,j}^{\min} A &:= \overline{F_{i,j} E} = [('F_i E \cap ''F_j E) + I]/I \cong ('F_i E \cap ''F_j E)/('F_i E \cap ''F_j E \cap I), \\ F_{i,j}^{\max} A &:= \overline{F_i E} \cap \overline{F_j E} = [(F_i E + I) \cap (F_j E + I)]/I = 'F_i A \cap ''F_j A. \end{aligned}$$

Clearly, for any i, j , one has $F_{i,j}^{\min} A \subset F_{i,j}^{\max} A$ where the inclusion is strict, in general. Therefore, writing $R^{\min} A$, resp. $R^{\max} A$, for the Rees module associated with the bifiltration $F_{i,j}^{\min} A$, resp. $F_{i,j}^{\max} A$, one obtains a canonical, not necessarily injective, bigraded $\mathbb{C}[\tau_1, \tau_2]$ -module homomorphism $\text{can} : R^{\min} A \rightarrow R^{\max} A$.

7.2. A flat scheme over \mathbb{C}^2 . We have the standard grading $\mathbb{C}[t] = \bigoplus_{i \geq 0} \mathbb{C}^i[t]$, resp. filtration $F_m = \mathbb{C}^{\leq m}[t] = \bigoplus_{i \leq m} \mathbb{C}^i[t]$, where $\mathbb{C}^i[t]$ denotes the space of degree i homogeneous polynomials on t . Similarly, one has a bigrading $\mathbb{C}[\mathfrak{T}] = \bigoplus_{i,j} \mathbb{C}^{i,j}[\mathfrak{T}]$ where $\mathbb{C}^{i,j}[\mathfrak{T}] := \mathbb{C}^i[t] \otimes \mathbb{C}^j[t]$. Associated with this bigrading, we have the pair of filtrations $'F \cdot \mathbb{C}[\mathfrak{T}]$ and $''F \cdot \mathbb{C}[\mathfrak{T}]$, respectively, and the corresponding bifiltration $F_{m,n} \mathbb{C}[\mathfrak{T}] = 'F_m \mathbb{C}[\mathfrak{T}] \cap ''F_n \mathbb{C}[\mathfrak{T}] = \bigoplus_{\{i \leq m, j \leq n\}} \mathbb{C}^{i,j}[\mathfrak{T}]$, $m, n \geq 0$, cf. §7.1.

Recall the setting of §1.7. Thus, we have the semisimple pair $\mathbf{h} = (h_1, h_2) \in \mathfrak{T}$ associated with the principal nilpotent pair $\mathbf{e} = (e_1, e_2)$. The coordinate ring of the finite subscheme $W \cdot \mathbf{h} \subset \mathfrak{T}$ has the form $\mathbb{C}[W \cdot \mathbf{h}] = \mathbb{C}[\mathfrak{T}]/I_{\mathbf{h}}$ where $I_{\mathbf{h}} \subset \mathbb{C}[\mathfrak{T}]$ is an ideal generated by the elements $\{f - f(\mathbf{h}), f \in \mathbb{C}[\mathfrak{T}]^W\}$. The quotient algebra $\mathbb{C}[W \cdot \mathbf{h}] = \mathbb{C}[\mathfrak{T}]/I_{\mathbf{h}}$ inherits a pair, $'F \cdot \mathbb{C}[W \cdot \mathbf{h}]$ and $''F \cdot \mathbb{C}[W \cdot \mathbf{h}]$ of quotient filtrations. Associated with these filtrations, there are bifiltrations $F_{i,j}^{\max} \mathbb{C}[W \cdot \mathbf{h}]$, resp. $F_{i,j}^{\min} \mathbb{C}[W \cdot \mathbf{h}]$.

We put $\mathfrak{Y} := \text{Spec } R^{\max} \mathbb{C}[W \cdot \mathbf{h}]$. This is an affine scheme that comes equipped with a $W \times \mathbb{C}^\times \times \mathbb{C}^\times$ -action and with a $\mathbb{C}^\times \times \mathbb{C}^\times$ -equivariant morphism $\wp : \mathfrak{Y} \rightarrow \mathbb{C}^2$ induced by the canonical algebra imbedding $\mathbb{C}[\tau_1, \tau_2] \hookrightarrow R^{\max} \mathbb{C}[W \cdot \mathbf{h}]$.

Let $\vartheta : \mathfrak{t} \rightarrow \mathfrak{t}/W$ be the quotient morphism and write $\mathbb{C}[\vartheta^{-1}(W \cdot h_2)]$ for the coordinate ring of the *scheme theoretic* fiber of ϑ over the orbit $W \cdot h_2$ viewed as a closed point of \mathfrak{t}/W .

Lemma 7.2.1. (i) *The scheme \mathfrak{Y} is a reduced, flat and finite scheme over \mathbb{C}^2 .*

There are natural W -equivariant algebra isomorphisms

$$\mathbb{C}[\wp^{-1}(0,0)] \cong \text{gr}^{\max} \mathbb{C}[W \cdot \mathbf{h}] = \bigoplus_{m,n \geq 0} \frac{F_{m,n}^{\max} \mathbb{C}[W \cdot \mathbf{h}]}{F_{m-1,n}^{\max} \mathbb{C}[W \cdot \mathbf{h}] + F_{m,n-1}^{\max} \mathbb{C}[W \cdot \mathbf{h}]}, \quad (7.2.2)$$

$$\mathbb{C}[\wp^{-1}(0,1)] \cong ' \text{gr} \mathbb{C}[W \cdot \mathbf{h}] \cong \mathbb{C}[\vartheta^{-1}(W \cdot h_2)], \quad (7.2.3)$$

$$\mathbb{C}[\wp^{-1}(1,1)] \cong \mathbb{C}[W \cdot \mathbf{h}]. \quad (7.2.4)$$

(ii) *We have $F_{\mathbf{d}_1, \mathbf{d}_2}^{\max} \mathbb{C}[W \cdot \mathbf{h}] = \mathbb{C}[W \cdot \mathbf{h}]$, where $\mathbf{d}_s = \#R_s^+$, $s = 1, 2$, see §1.7.*

Proof. The isomorphism in (7.2.2) follows from (7.1.2), resp. the isomorphism in (7.2.4) follows from definitions. The latter isomorphism implies, by $\mathbb{C}^\times \times \mathbb{C}^\times$ -equivariance, that we have $\wp^{-1}(\mathbb{C}^\times \times \mathbb{C}^\times) \cong \mathbb{C}^\times \times \mathbb{C}^\times \times W \cdot \mathbf{h}$.

Further, the map \wp is flat by Corollary 7.1.5. Therefore, \mathfrak{Y} is a Cohen-Macaulay scheme such that $\wp^{-1}(\mathbb{C}^\times \times \mathbb{C}^\times)$ is a reduced scheme. It follows that the scheme \mathfrak{Y} is reduced.

To complete the proof of the lemma, we consider the Levi subalgebra $\mathfrak{g}^1 = \mathfrak{g}_{h_2}$ and its Weyl group W_1 , the subgroup of W generated by reflections with respect to the set $R_1^+ \subset \mathfrak{t}^*$ formed by the roots $\alpha \in R^+$ such that $\alpha(h_2) = 0$.

Let $I_1 \subset \mathbb{C}[\mathfrak{t}]$ denote the ideal of the orbit $W_1 \cdot h_1 \subset \mathfrak{t}$ viewed as a reduced finite subscheme of \mathfrak{t} . The element h_1 has trivial isotropy group under the W_1 -action, [Gi, Proposition 3.2]. Therefore, a standard argument based on the fact that $\mathbb{C}[\mathfrak{t}]$ is a free $\mathbb{C}[\mathfrak{t}]^{W_1}$ -module shows that the ideal $\text{gr}^F I_1$ equals $(\mathbb{C}[\mathfrak{t}]_+^{W_1})$, the ideal generated by W_1 -invariant homogeneous polynomials of positive degree. We deduce a chain of graded algebra isomorphisms

$$\text{gr}^F \mathbb{C}[W_1 \cdot h_1] \cong \text{gr}^F \mathbb{C}[\mathfrak{t}] / \text{gr}^F I_1 = \mathbb{C}[\mathfrak{t}] / (\mathbb{C}[\mathfrak{t}]_+^{W_1}) = \mathbb{C}[(\vartheta_1)^{-1}(0)] \quad (7.2.5)$$

where we have used the notation $\vartheta_1 : \mathfrak{t} \rightarrow \mathfrak{t}/W_1$ for the quotient morphism. Thus, there is an isomorphism $\text{Spec } \text{gr}^F \mathbb{C}[W_1 \cdot h_1] \cong (\vartheta_1)^{-1}(0)$ of (not necessarily reduced) schemes.

Let $pr_2 : W \cdot \mathbf{h} \rightarrow W \cdot h_2$, $w(\mathbf{h}) \mapsto w(h_2)$ be the second projection. We may view the fiber $pr_2^{-1}(t)$, $t \in W \cdot h_2$, as a subset of $\mathfrak{t} \times \{t\} \cong \mathfrak{t}$. Fix $m \geq 0$ and let $f \in {}'F_m \mathbb{C}[W \cdot \mathbf{h}]$. Then, by definition of the filtration $'F \cdot \mathbb{C}[W \cdot \mathbf{h}]$, for any $t \in W \cdot h_2$, there exists a polynomial $\tilde{f}_t \in \mathbb{C}^{\leq m}[\mathfrak{t}]$ such that we have $\tilde{f}_t|_{pr_2^{-1}(t)} = f|_{pr_2^{-1}(t)}$. Conversely, let $f \in \mathbb{C}[W \cdot \mathbf{h}]$ be an element such that, for any $t \in W \cdot h_2$, there exists a polynomial $\tilde{f}_t \in \mathbb{C}^{\leq m}[\mathfrak{t}]$ such that $\tilde{f}_t|_{pr_2^{-1}(t)} = f|_{pr_2^{-1}(t)}$. Then, using Langrange interpolation formula, one shows that $f \in {}'F_m \mathbb{C}[W \cdot \mathbf{h}]$. Thus, we have established a natural $W \times \mathbb{C}^\times$ -equivariant isomorphism

$$\text{Spec}({}'\text{gr } \mathbb{C}[W \cdot \mathbf{h}]) \cong W \times_{W_1} (\text{Spec } \text{gr}^F \mathbb{C}[pr_2^{-1}(h_2)]),$$

of (not necessarily reduced) schemes.

Note further that we have $pr_2^{-1}(h_2) = W_1 \cdot h_1$. Moreover, the scheme $\text{Spec } \text{gr}^F \mathbb{C}[W_1 \cdot h_1]$ is isomorphic to $(\vartheta_1)^{-1}(0)$ thanks to (7.2.5). Thus, one obtains $W \times \mathbb{C}^\times$ -equivariant isomorphisms

$$\text{Spec}({}'\text{gr } \mathbb{C}[W \cdot \mathbf{h}]) \cong W \times_{W_1} ((\vartheta_1)^{-1}(0)) \cong \vartheta^{-1}(W \cdot h_2). \quad (7.2.6)$$

The isomorphisms in (7.2.3) follow since we have

$$\mathbb{C}[\wp^{-1}(0, 1)] = \frac{\text{R}^{\max} \mathbb{C}[W \cdot \mathbf{h}]}{\tau_1 \cdot \text{R}^{\max} \mathbb{C}[W \cdot \mathbf{h}] + (\tau_2 - 1) \cdot \text{R}^{\max} \mathbb{C}[W \cdot \mathbf{h}]} = {}'\text{gr } \mathbb{C}[W \cdot \mathbf{h}] = \mathbb{C}[\vartheta^{-1}(W \cdot h_2)].$$

Here, the first isomorphism holds by definition, the second isomorphism is (7.1.3), and the last isomorphism is (7.2.6).

To prove part (ii), we recall that the coinvariant algebra $\mathbb{C}[\mathfrak{t}] / (\mathbb{C}[\mathfrak{t}]_+^{W_1})$ is a graded algebra which is known to be concentrated in degrees $\leq \mathbf{d}_1$. We deduce, using isomorphisms (7.2.5) and (7.2.6), that the graded algebra $'\text{gr } \mathbb{C}[W \cdot \mathbf{h}]$ is also concentrated in degrees $\leq \mathbf{d}_1$. Thus, we have $'F_{\mathbf{d}_1} \mathbb{C}[W \cdot \mathbf{h}] = \mathbb{C}[W \cdot \mathbf{h}]$. By symmetry, we get $'F_{\mathbf{d}_2} \mathbb{C}[W \cdot \mathbf{h}] = \mathbb{C}[W \cdot \mathbf{h}]$. Part (ii) of the lemma follows. \square

The group $\mathbb{C}^\times \times \mathbb{C}^\times$ acts on \mathbb{C}^2 and on \mathfrak{T} . This makes $\mathbb{C}[\mathbb{C}^2 \times \mathfrak{T}] = \mathbb{C}[\tau_1, \tau_2] \otimes \mathbb{C}[\mathfrak{T}]$ a bigraded algebra with respect to the natural bigrading on a tensor product. Let $I_{\mathfrak{h}} = \bigoplus_{i,j} I_{\mathfrak{h}}^{i,j} \subset \mathbb{C}[\tau_1, \tau_2] \otimes \mathbb{C}[\mathfrak{T}]$ be a bihomogeneous ideal generated by the set

$$\{\tau_1^m \tau_2^n \cdot f_{m,n}(\mathbf{h}) - f_{m,n} \mid f_{m,n} \in (\mathbb{C}^{m,n}[\mathfrak{T}])^W, m, n \geq 0\},$$

of bihomogeneous elements. It is clear that we have $\mathbb{C}[\mathbb{C}^2 \times \mathfrak{T}] / I_{\mathfrak{h}} = \mathbb{C}[\mathbb{C}^2 \times_{\mathfrak{T}/W} \mathfrak{T}]$, where the fiber product on the right involves the map $\mathbb{C}^2 \rightarrow \mathfrak{T}/W$, $(\tau_1, \tau_2) \mapsto (\tau_1 \cdot h_1, \tau_2 \cdot h_2) \bmod W$.

According to the definition of the map \aleph of Lemma 7.1.4, for any $f_{m,n} \in \mathbb{C}^{m,n}[\mathfrak{T}]$, we find $\aleph(\tau_1^m \tau_2^n \cdot f_{m,n}(\mathbf{h}) - f_{m,n}) = \tau_1^m \tau_2^n \cdot (f_{m,n}(\mathbf{h}) - f_{m,n})$. The right hand side here clearly belongs to the subspace $\tau_1^m \tau_2^n \cdot (I_{\mathbf{h}} \cap F_{m,n} \mathbb{C}[\mathfrak{T}])$. Hence, for any $i, j \geq 0$, one has an inclusion

$\aleph(I_{\mathfrak{h}}^{i,j}) \subset \tau_1^i \tau_2^j \cdot (I_{\mathfrak{h}} \cap F_{i,j} \mathbb{C}[\mathfrak{T}])$. These inclusions insure that the map \aleph descends to a well defined map $\aleph : \mathbb{C}[\mathbb{C}^2 \times \mathfrak{T}]/I_{\mathfrak{h}} \rightarrow \mathbb{R}\mathbb{C}[\mathfrak{T}]/RI_{\mathfrak{h}} = \mathbb{R}^{\min} \mathbb{C}[W \cdot \mathfrak{h}]$.

Thus, we obtain the following chain of bigraded W -equivariant $\mathbb{C}[\tau_1, \tau_2]$ -algebra maps

$$\mathbb{C}[\mathbb{C}^2 \times_{\mathfrak{T}/W} \mathfrak{T}] = \mathbb{C}[\mathbb{C}^2 \times \mathfrak{T}]/I_{\mathfrak{h}} \xrightarrow{\aleph} \mathbb{R}^{\min} \mathbb{C}[W \cdot \mathfrak{h}] \xrightarrow{\text{can}} \mathbb{R}^{\max} \mathbb{C}[W \cdot \mathfrak{h}] = \mathbb{C}[\mathfrak{Y}]. \quad (7.2.7)$$

7.3. A 2-parameter deformation of \mathfrak{e} . It follows from definitions that the nilpotent elements e_1, e_2 and the semisimple elements h_1, h_2 satisfy the following commutation relations, cf. [Gi, Theorem 1.2]

$$[e_1, e_2] = 0 = [h_1, h_2], \quad [h_i, e_j] = \delta_{ij} \cdot e_i, \quad i, j \in \{1, 2\}. \quad (7.3.1)$$

The above relations imply that the elements $e_1 + \tau_1 \cdot h_1$ and $e_2 + \tau_2 \cdot h_2$ commute for any $\tau_1, \tau_2 \in \mathbb{C}$. Therefore, one can define the following map that will play an important role in the arguments below

$$\varkappa : \mathbb{C}^2 \rightarrow \mathfrak{C}, \quad (\tau_1, \tau_2) \mapsto \varkappa(\tau_1, \tau_2) = (e_1 + \tau_1 \cdot h_1, e_2 + \tau_2 \cdot h_2). \quad (7.3.2)$$

Lemma 7.3.3. *For any $f \in \mathbb{C}[\mathfrak{C}]^G$ and $\tau_1, \tau_2 \in \mathbb{C}$, one has $f(\varkappa(\tau_1, \tau_2)) = f(\tau_1 \cdot h_1, \tau_2 \cdot h_2)$.*

Proof. This easily follows from Lemma 6.1.1. An alternative way of proving the lemma is based on a useful formula $\exp \text{ad}(\tau \cdot e_i)(h_j) = h_j - \delta_{ij} \cdot \tau \cdot e_i$, $i = 1, 2$. Define a holomorphic (non-algebraic) map $\gamma : \mathbb{C}^\times \times \mathbb{C}^\times \rightarrow G$ as follows $\gamma(\tau_1, \tau_2) = \exp(\frac{1}{\tau_1} e_1 + \frac{1}{\tau_2} e_2)$. Then, using the formula, we find (we note that $h_i = 0$ holds only if $e_i = 0$):

$$\tau \cdot h_i = \text{Ad} \exp(\frac{1}{\tau} e_i)(e_i + \tau \cdot h_i) \quad \forall \tau \neq 0, i = 1, 2; \quad (7.3.4)$$

$$\text{hence, we have } (\tau_1 \cdot h_1, \tau_2 \cdot h_2) = \text{Ad} \gamma(\tau_1, \tau_2)(\varkappa(\tau_1, \tau_2)), \quad (\tau_1, \tau_2) \in \mathbb{C}^\times \times \mathbb{C}^\times.$$

The last equation in (7.3.4) clearly implies the lemma. \square

From Lemma 7.3.3 we deduce that, for any $(\tau_1, \tau_2) \in \mathbb{C}^2$, in $\mathfrak{C} // G = \mathfrak{T} / W$ one has $\varkappa(\tau_1, \tau_2) \bmod G = (\tau_1 \cdot h_1, \tau_2 \cdot h_2) \bmod W$. Thus, one can introduce a map $\tilde{\varkappa}$ that fits into the following diagram of cartesian squares

$$\begin{array}{ccccc} \mathbb{C}^2 \times_{\mathfrak{T}/W} \mathfrak{T} & \stackrel{\sim}{=} & \mathbb{C}^2 \times_{\mathfrak{C}} (\mathfrak{C} \times_{\mathfrak{C} // G} \mathfrak{T}) & \xrightarrow{\tilde{\varkappa}} & \mathfrak{C} \times_{\mathfrak{C} // G} \mathfrak{T} & \xrightarrow{pr_2} & \mathfrak{T} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbb{C}^2 & \xrightarrow{\varkappa} & \mathfrak{C} & \xrightarrow{proj} & \mathfrak{C} // G & = & \mathfrak{T} / W. \end{array}$$

The composite of the maps in (7.2.7) is a homomorphism of coordinate rings that gives a certain $W \times \mathbb{C}^\times \times \mathbb{C}^\times$ -equivariant morphism $\mathfrak{Y} \rightarrow \mathbb{C}^2 \times_{\mathfrak{T}/W} \mathfrak{T}$. Thus, we obtain the following chain of $W \times \mathbb{C}^\times \times \mathbb{C}^\times$ -equivariant morphisms

$$\mathfrak{Y} \longrightarrow \mathbb{C}^2 \times_{\mathfrak{T}/W} \mathfrak{T} \stackrel{\sim}{=} \mathbb{C}^2 \times_{\mathfrak{C}} (\mathfrak{C} \times_{\mathfrak{C} // G} \mathfrak{T}) \xrightarrow{\tilde{\varkappa}} \mathfrak{C} \times_{\mathfrak{C} // G} \mathfrak{T}. \quad (7.3.5)$$

The composite morphism in (7.3.5) factors through the reduced scheme \mathfrak{X} , since the scheme \mathfrak{Y} is reduced by Lemma 7.2.1. Thus, we have constructed a $W \times \mathbb{C}^\times \times \mathbb{C}^\times$ -equivariant morphism Υ that fits into a commutative diagram

$$\begin{array}{ccc} \mathfrak{Y} & \xrightarrow{\Upsilon} & \mathfrak{X} \\ \downarrow \wp & & \downarrow p \\ \mathbb{C}^2 & \xrightarrow{\varkappa} & \mathfrak{C}_{\text{red}} \end{array} \quad (7.3.6)$$

The proof of Theorem 1.7.2 is based on the following result concerning the structure of the map Υ over $\mathbb{C}^2 \setminus \{0\}$, the complement of the origin $0 = (0, 0) \in \mathbb{C}^2$.

Proposition 7.3.7. (i) *The image of the map \varkappa is contained in \mathfrak{C}^r .*

(ii) *For any $(x, y) \in \varkappa(\mathbb{C}^2 \setminus \{0\})$ we have:*

- *The fiber $p^{-1}(x, y)$ is contained in the smooth locus of the variety \mathfrak{X}*
- *The map p is flat over (x, y) .*

(iii) *The map $\wp \times \Upsilon$ yields the following isomorphism of schemes over $\mathbb{C}^2 \setminus \{0\}$:*

$$\wp^{-1}(\mathbb{C}^2 \setminus \{0\}) \xrightarrow[\sim]{\wp \times \Upsilon} (\mathbb{C}^2 \setminus \{0\}) \times_{\mathfrak{C}} \mathfrak{X}.$$

Proof. Part (i) is clear for $(\tau_1, \tau_2) = (0, 0)$ since we have $\varkappa(0) = (e_1, e_2) = \mathbf{e} \in \mathfrak{C}^r$. Thus, for the rest of the proof we may assume that $(x, y) = \varkappa(\tau_1, \tau_2)$ for some $(\tau_1, \tau_2) \in \mathbb{C}^2 \setminus \{0\}$.

One has an open covering $\mathbb{C}^2 \setminus 0 = U_1 \cup U_2$ where $U_i = \{(\tau_1, \tau_2) \in \mathbb{C}^2 \mid \tau_i \neq 0\}$. We will consider the case where $(\tau_1, \tau_2) \in U_2$, the other case being totally similar. Thus, we have $(x, y) = (e_1 + \tau_1 \cdot h_1, e_2 + \tau_2 \cdot h_2)$ for some $\tau_2 \neq 0$.

If $\tau_1 \neq 0$ then the element (x, y) is G -conjugate to the element $(\tau_1 \cdot h_1, \tau_2 \cdot h_2)$, by formula (7.3.4). In that case, one has $(x, y) \in \mathfrak{C}^{rs}$ and all the statements of parts (i)-(ii) of Proposition 7.3.7 are clear.

It remains to consider the case $\tau_1 = 0$. Thus, we have $(x, y) = (e_1, e_2 + \tau_2 \cdot h_2)$ where $\tau_2 \neq 0$. Applying the formula in the first line of (7.3.4), we deduce that the element (x, y) is G -conjugate to the element $(e_1, \tau_2 \cdot h_2)$. Further, using the \mathbb{C}^\times -action, we may assume without loss of generality that $\tau_2 = 1$.

To complete the proof of parts (i)-(ii), we write $\mathfrak{G} = \mathbb{C}^2 \otimes \mathfrak{g}$, resp. $\mathfrak{G} \times \mathfrak{T} = \mathbb{C}^2 \otimes (\mathfrak{g} \times \mathfrak{t})$. The natural GL_2 -action on \mathbb{C}^2 induces, via the action on the first tensor factor, a GL_2 -action on \mathfrak{G} , resp. on $\mathfrak{G} \times \mathfrak{T}$, such that the $\mathbb{C}^\times \times \mathbb{C}^\times$ -action considered earlier corresponds to the action of the maximal torus in GL_2 formed by diagonal matrices. The scheme $\mathfrak{C} \subset \mathfrak{G}$, resp. $\mathfrak{X} \subset \mathfrak{G} \times \mathfrak{T}$, is clearly GL_2 -stable. Further, it is clear that there is an element $g \in GL_2$ that takes the point $(e_1, h_2) \in \mathfrak{C}$ to the point $(e_1 + h_2, h_2)$. Now, according to [Gi, Proposition 3.2.3], the element e_1 is a principal nilpotent in the Levi subalgebra \mathfrak{g}_{h_2} . It follows that $e_1 + h_2$ is a regular element of \mathfrak{g} . Hence, we have $(e_1 + h_2, h_2) \in \mathfrak{C}_1$, cf. Definition 2.6.3. It follows that $(e_1 + h_2, h_2)$ is a smooth point of \mathfrak{C} . Moreover, the fiber $p^{-1}(e_1 + h_2, h_2)$ is contained in the smooth locus of \mathfrak{X} and the map p is flat over the point $(e_1 + h_2, h_2)$, by Lemma 2.1.5. All statements of Proposition 7.3.7(i)-(ii) follow from this using the GL_2 -action.

We see from part (ii) that $U_2 \times_{\mathfrak{C}} \mathfrak{X}$ is a reduced scheme, flat over U_2 . Proving part (iii) amounts to showing that the map $\wp^{-1}(U_2) \rightarrow U_2 \times_{\mathfrak{C}} \mathfrak{X}$ induced by $\wp \times \Upsilon$ is an isomorphism. The map in question is a $\mathbb{C}^\times \times \mathbb{C}^\times$ -equivariant morphism of flat schemes over $U_2 = \mathbb{C} \times \mathbb{C}^\times$. Therefore, this map is an isomorphism if and only if it induces an isomorphism

$$\Psi : \wp^{-1}(0, 1) \xrightarrow{\sim} \{(0, 1)\} \times_{\mathfrak{C}} \mathfrak{X} = p^{-1}(e_1, e_2 + h_2), \quad (7.3.8)$$

of the corresponding fibers over a single point $(0, 1) \in \mathbb{C} \times \mathbb{C}^\times$.

Observe next that the argument used in the proof of part (ii), based on the GL_2 -action, yields an isomorphism of schemes $p^{-1}(e_1, e_2 + h_2) \cong p^{-1}(e_1 + h_2, h_2)$. Further, we know that $e_1 + h_2$ is a regular element of \mathfrak{g} . Therefore, taking the fibers at $e_1 + h_2$ of the locally free sheaves involved in the chain of isomorphism (6.5.3), yields an algebra isomorphism $\mathbb{C}[p^{-1}(e_1 + h_2, h_2)] = \mathcal{R}_{(e_1 + h_2, h_2)} \cong \mathbb{C}[\vartheta^{-1}(W \cdot h_2)]$. On the other hand, by Lemma 7.2.1, we have $\mathbb{C}[\wp^{-1}(0, 1)] \cong \mathbb{C}[\vartheta^{-1}(W \cdot h_2)]$. Thus, we obtain the following chain of W -equivariant algebra isomorphisms

$$\mathbb{C}[\{(0, 1)\} \times_{\mathfrak{C}} \mathfrak{X}] = \mathbb{C}[p^{-1}(e_1, e_2 + h_2)] \cong \mathbb{C}[p^{-1}(e_1 + h_2, h_2)] \cong \mathbb{C}[\vartheta^{-1}(W \cdot h_2)] \cong \mathbb{C}[\wp^{-1}(0, 1)].$$

We claim that the composite of the above isomorphisms is equal to the algebra map $\Psi^* : \mathbb{C}[\{(0,1)\} \times_{\mathfrak{C}} \mathfrak{X}] \rightarrow \mathbb{C}[\wp^{-1}(0,1)]$ induced by the morphism Ψ in (7.3.8). To see this, one observes that the algebra $\mathbb{C}[\{(0,1)\} \times_{\mathfrak{C}} \mathfrak{X}]$ is a quotient of the algebra $\mathbb{C}[\mathfrak{T}]$. Hence, it suffices to verify our claim for linear functions on \mathfrak{T} , the generators of the algebra $\mathbb{C}[\mathfrak{T}]$. Checking the latter is straightforward and is left to the reader.

We conclude that the morphism Ψ in (7.3.8) is itself an isomorphism, and part (iii) of Proposition 7.3.7 follows. \square

7.4. Proof of Theorem 1.7.2. Part (i) of Proposition 7.3.7 implies that the map \varkappa gives a closed imbedding $\varkappa : \mathbb{C}^2 \hookrightarrow \mathfrak{C}^r$. Therefore, we may view \varkappa as a map $\mathbb{C}^2 \rightarrow \mathfrak{C}_{\text{norm}}$ and form an associated fiber product $\mathbb{C}^2 \times_{\mathfrak{C}^r} \mathfrak{X}_{\text{norm}}$. From diagram (7.3.6) we obtain, by base change, the following commutative diagram of $W \times \mathbb{C}^\times \times \mathbb{C}^\times$ -equivariant morphisms of schemes over \mathbb{C}^2

$$\begin{array}{ccccc} \mathfrak{Y} & \xrightarrow{\wp \times \Upsilon} & \mathbb{C}^2 \times_{\mathfrak{C}^r} \mathfrak{X} & \xleftarrow{\text{Id}_{\mathbb{C}^2} \times \psi} & \mathbb{C}^2 \times_{\mathfrak{C}^r} \mathfrak{X}_{\text{norm}} \\ \downarrow \wp & & \downarrow & & \downarrow p_\varkappa \\ \mathbb{C}^2 & \xlongequal{\text{Id}} & \mathbb{C}^2 & \xlongequal{\text{Id}} & \mathbb{C}^2. \end{array} \quad (7.4.1)$$

Restricting this diagram to the open set $\mathbb{C}^2 \setminus \{0\}$ one obtains a diagram of isomorphisms

$$\wp^{-1}(\mathbb{C}^2 \setminus \{0\}) \xrightarrow[\sim]{\wp \times \Upsilon} (\mathbb{C}^2 \setminus \{0\}) \times_{\mathfrak{C}^r} \mathfrak{X} \xleftarrow[\sim]{\text{Id}_{\mathbb{C}^2} \times \psi} (\mathbb{C}^2 \setminus \{0\}) \times_{\mathfrak{C}^r} \mathfrak{X}_{\text{norm}}. \quad (7.4.2)$$

Here the map $\wp \times \Upsilon$ on the left is an isomorphism thanks to part (iii) of Proposition 7.3.7 and the map $\text{Id}_{\mathbb{C}^2} \times \psi$ on the right induced by the normalization map $\psi : \mathfrak{X}_{\text{norm}} \rightarrow \mathfrak{X}$ is an isomorphism thanks to part (ii) of Proposition 7.3.7.

The map $p_{\text{norm}} : \mathfrak{X}_{\text{norm}} \rightarrow \mathfrak{C}_{\text{norm}}$ being flat, by flat base change we get that $(p_\varkappa)_* \mathcal{O}_{\mathbb{C}^2 \times_{\mathfrak{C}^r} \mathfrak{X}_{\text{norm}}} \cong \varkappa^*[(p_{\text{norm}})_* \mathcal{O}_{\mathfrak{X}_{\text{norm}}}] = \varkappa^* \mathcal{R}$ is a locally free sheaf on \mathbb{C}^2 . We conclude that $\wp_* \mathcal{O}_{\mathfrak{Y}}$ and $\varkappa^* \mathcal{R}$ are $W \times \mathbb{C}^\times \times \mathbb{C}^\times$ -equivariant locally free sheaves of $\mathcal{O}_{\mathbb{C}^2}$ -algebras and we have a $W \times \mathbb{C}^\times \times \mathbb{C}^\times$ -equivariant algebra isomorphism $(\wp_* \mathcal{O}_{\mathfrak{Y}})|_{\mathbb{C}^2 \setminus \{0\}} \cong (\varkappa^* \mathcal{R})|_{\mathbb{C}^2 \setminus \{0\}}$ induced by diagram (7.4.2). Therefore, the isomorphism can be extended (uniquely) across the origin $0 \in \mathbb{C}^2$. The resulting isomorphism $\varkappa^* \mathcal{R} \xrightarrow{\sim} \wp_* \mathcal{O}_{\mathfrak{Y}}$ is automatically $W \times \mathbb{C}^\times \times \mathbb{C}^\times$ -equivariant and respects the $\mathcal{O}_{\mathbb{C}^2}$ -algebra structures. Restricting the latter isomorphism to the fibers at the origin yields a bigraded W -equivariant algebra isomorphism $\mathcal{R}_e \xrightarrow{\sim} \mathbb{C}[\wp^{-1}(0)]$. On the other hand, by (7.2.2), we have $\mathbb{C}[\wp^{-1}(0)] = \text{gr}^{\max} \mathbb{C}[W \cdot \mathfrak{h}]$. The isomorphism of Theorem 1.7.2 follows.

The last claim in the theorem is a consequence of Lemma 7.2.1(ii). \square

7.5. Proof of Theorem 1.8.1. First of all, we introduce some notation and define several vector spaces associated with the pair $'F, \mathbb{C}[\mathfrak{T}]$ and $''F, \mathbb{C}[\mathfrak{T}]$ of filtrations on $\mathbb{C}[\mathfrak{T}]$.

We may (and will) view elements of the vector space $\text{Sym } \mathfrak{T}$ as constant coefficient differential operators on $\mathfrak{T} = \mathfrak{t} \times \mathfrak{t}$. Thus, an element $u \in \text{Sym}^i \mathfrak{t} \otimes \text{Sym}^j \mathfrak{t}$ is a bihomogeneous differential operator of order i with respect to the first, resp. of order j with respect to the second, factor in the cartesian product $\mathfrak{t} \times \mathfrak{t} = \mathfrak{T}$. For any $m \geq 0$, we write $S^m := \text{Sym}^m \mathfrak{t}$ and let $\widehat{S} := \prod_{i,j \geq 0} (\text{Sym}^i \mathfrak{t} \otimes \text{Sym}^j \mathfrak{t})$.

For any $i, j \geq 0$, the assignment $u \times f \mapsto u(f)(0)$ gives a perfect pairing $(S^i \otimes S^j) \times \mathbb{C}^{i,j}[\mathfrak{T}] \rightarrow \mathbb{C}$. We obtain canonical isomorphisms

$$(\mathbb{C}[\mathfrak{T}])^* = \left(\bigoplus_{i,j} \mathbb{C}^{i,j}[\mathfrak{T}] \right)^* \cong \prod_{i,j \geq 0} (\mathbb{C}^{i,j}[\mathfrak{T}])^* = \prod_{i,j \geq 0} (S^i \otimes S^j) = \widehat{S}. \quad (7.5.1)$$

Thus, one has a perfect pairing $\widehat{S} \times \mathbb{C}[\mathfrak{T}] \rightarrow \mathbb{C}$. It is instructive to think of an element $u \in \widehat{S}$ as a constant coefficient differential operator on \mathfrak{T} "of infinite order". Accordingly, we write the above pairing as $u \times f \mapsto u(f)(0)$ similarly to the finite order case. Given a vector subspace $H \subset \mathbb{C}[\mathfrak{T}]$, let $H^\perp \subset \widehat{S}$ denote the annihilator of H with respect the pairing.

Below, we will use the following simplified notation $'F_i := 'F_i \mathbb{C}[\mathfrak{T}]$, resp. $''F_j := ''F_j \mathbb{C}[\mathfrak{T}]$, and $F_{i,j} = F_{i,j} \mathbb{C}[\mathfrak{T}] = 'F_i \cap ''F_j$. Further, let $I = I_{\mathbf{h}}$ so, we have $\mathbb{C}[\mathfrak{T}]/I = \mathbb{C}[W \cdot \mathbf{h}]$. Then, for any $m, n \geq 0$, there are natural imbeddings

$$F_{m,n} = 'F_m \cap ''F_n \hookrightarrow ['F_m + I] \cap [''F_n + I] \hookrightarrow 'F_m + ''F_n + I \hookrightarrow \mathbb{C}[\mathfrak{T}].$$

These imbeddings induce the following chain of natural linear maps

$$\begin{aligned} \text{gr}_{m,n} \mathbb{C}[\mathfrak{T}] &= \frac{F_{m,n}}{F_{m,n-1} + F_{m-1,n}} = \frac{'F_m \cap ''F_n}{'F_{m-1} \cap ''F_n + 'F_m \cap ''F_{n-1}} \\ &\xrightarrow{a} \text{gr}_{m,n}^{\max} \mathbb{C}[W \cdot \mathbf{h}] = \frac{['F_m + I] \cap [''F_n + I]}{['F_{m-1} + I] \cap [''F_n + I] + ['F_m + I] \cap [''F_{n-1} + I]} \\ &\xrightarrow{b} \frac{\mathbb{C}[\mathfrak{T}]}{'F_{m-1} + ''F_{n-1} + I}. \quad (7.5.2) \end{aligned}$$

For each $m, n \geq 0$, let

$$\widehat{S}^{>m} := \prod_{\{i>m, j \geq 0\}} S^i \otimes S^j, \quad \text{resp.} \quad ''\widehat{S}^{>n} := \prod_{\{i \geq 0, j>n\}} S^i \otimes S^j.$$

Clearly, we have $('F_i)^\perp = \widehat{S}^{>i}$, resp. $(''F_j)^\perp = ''\widehat{S}^{>j}$. Hence, we get $F_{i,j}^\perp = ('F_i \cap ''F_j)^\perp = \widehat{S}^{>i} + ''\widehat{S}^{>j}$ and $['F_i + I]^\perp = \widehat{S}^{>i} \cap I^\perp$, resp. $[''F_j + I]^\perp = ''\widehat{S}^{>j} \cap I^\perp$. Further, we find

$$\begin{aligned} \left(\frac{\mathbb{C}[\mathfrak{T}]}{'F_{i-1} + ''F_{j-1} + I} \right)^* &= ('F_{i-1} + ''F_{j-1} + I)^\perp = \widehat{S}^{>i-1} \cap ''\widehat{S}^{>j-1} \cap I^\perp; \\ \left(\frac{F_{m,n}}{F_{m,n-1} + F_{m-1,n}} \right)^* &= \frac{(\widehat{S}^{>m-1} + ''\widehat{S}^{>n}) \cap (\widehat{S}^{>m} + ''\widehat{S}^{>n-1})}{\widehat{S}^{>m} + ''\widehat{S}^{>n}} = S^m \otimes S^n. \quad (7.5.3) \end{aligned}$$

Therefore, dualizing the maps in (7.5.2) one gets the following linear maps

$$\widehat{S}^{>m-1} \cap ''\widehat{S}^{>n-1} \cap I^\perp \xrightarrow{b^*} (\text{gr}_{m,n}^{\max} \mathbb{C}[W \cdot \mathbf{h}])^* \xrightarrow{a^*} (\text{gr}_{m,n} \mathbb{C}[\mathfrak{T}])^* = S^m \otimes S^n. \quad (7.5.4)$$

Proof of Theorem 1.8.1. We write $\check{\alpha} \in \mathfrak{t}$ for the coroot corresponding to a root $\alpha \in R^+$. Recall the notation $R_s^+ \subset R^+$ for the set of positive roots of the Levi subalgebra \mathfrak{g}^s , $s = 1, 2$, and put $\delta_s := \prod_{\alpha \in R_s^+} \check{\alpha} \in S^{\mathbf{d}_s}$. Further, let $V_s = \mathbb{C}[W] \cdot \delta_s \subset S^{\mathbf{d}_s}$, $s = 1, 2$, be the W -submodule generated by the element δ_s . It is known, that V_s is a simple W -module and, moreover, this W -module occurs in $S^{\mathbf{d}_s}$ with multiplicity one, see [Mc]. Therefore, there is a canonically defined copy of the simple $W \times W$ -submodule $V_1 \otimes V_2$ inside $S^{\mathbf{d}_1} \otimes S^{\mathbf{d}_2}$. Dually, there is a canonically defined copy of the simple $W \times W$ -submodule $\check{V}_1 \otimes \check{V}_2$ inside $\mathbb{C}^{\mathbf{d}_1, \mathbf{d}_2}[\mathfrak{T}]$ where $\check{V}_s \subset \mathbb{C}^{\mathbf{d}_s}[\mathfrak{t}]$, $s = 1, 2$, stands for the contragredient W -module (in fact, one has $V_s \cong \check{V}_s$, since any simple W -module is known to be selfdual).

Following [Gi, §4], we consider the element $\nabla := \sum_{w \in W} \text{sign}(w) \cdot e^{w(\mathbf{h})} \in \widehat{S}$. More explicitly, we let

$$\nabla = \sum_{i,j \geq 0} \nabla_{i,j} \quad \text{where} \quad \nabla_{i,j} := \frac{1}{i! \cdot j!} \sum_{w \in W} \text{sign}(w) \cdot w(h_1^i \otimes h_2^j) \in S^i \otimes S^j. \quad (7.5.5)$$

Using the Taylor formula, for any polynomial f on \mathfrak{T} , we find $e^{w(\mathbf{h})}(f)(0) = f(w(\mathbf{h}))$. Hence, we get $\nabla(f)(0) = \sum_{w \in W} \text{sign}(w) \cdot f(w(\mathbf{h}))$. It follows that the linear function $\mathbb{C}[\mathfrak{T}] \rightarrow \mathbb{C}$, $f \mapsto \nabla(f)(0)$ annihilates the ideal $I = I_{\mathbf{h}}$ in other words, we have $\nabla \in I^\perp$. Further, by [Gi, Lemma 4.3], one has $\nabla_{i,j} = 0$ whenever $i < \mathbf{d}_1$ or $j < \mathbf{d}_2$. We conclude that $\nabla \in \widehat{S}^{>\mathbf{d}_1-1} \cap {}''\widehat{S}^{>\mathbf{d}_2-1} \cap I^\perp$. Thus, there is a well defined element $a^*(b^*(\nabla)) \in S^{\mathbf{d}_1} \otimes S^{\mathbf{d}_2}$, cf. (7.5.4). In addition, it is clear from (7.5.3)-(7.5.5) that we have an equation:

$$\nabla_{\mathbf{d}_1, \mathbf{d}_2} = a^*(b^*(\nabla)) \bmod (\widehat{S}^{>\mathbf{d}_1} + {}''\widehat{S}^{>\mathbf{d}_2}). \quad (7.5.6)$$

From now on, we assume that \mathbf{e} is a *non-exceptional* principal nilpotent pair. Then, according to [Gi, Theorem 4.4], one has an isomorphism $V_2 \cong V_1 \otimes \text{sign}$ of W -modules. It follows that $(V_1 \otimes V_2)^{\text{sign}}$ is a 1-dimensional vector space, moreover, according to *loc cit*, the element $\nabla_{\mathbf{d}_1, \mathbf{d}_2}$ is a nonzero element of that vector space. Dually, $(\check{V}_1 \otimes \check{V}_2)^{\text{sign}}$ is a 1-dimensional vector space and $\Delta_{\mathbf{e}}$ is a nonzero element of that vector space.

The canonical perfect pairing $(V_1 \otimes V_2) \times (\check{V}_1 \otimes \check{V}_2) \rightarrow \mathbb{C}$ is W -invariant with respect to the W -diagonal action. Therefore, this pairing yields a perfect pairing between $(V_1 \otimes V_2)^{\text{sign}}$ and $(\check{V}_1 \otimes \check{V}_2)^{\text{sign}}$, the corresponding sign-isotypic components. These are 1-dimensional vector spaces, with $\Delta_{\mathbf{e}}$ and $\nabla_{\mathbf{d}_1, \mathbf{d}_2}$ being respective base vectors. Hence, one must have $\nabla_{\mathbf{d}_1, \mathbf{d}_2}(\Delta_{\mathbf{e}})(0) \neq 0$. Thus, writing $\langle\langle -, - \rangle\rangle : (\text{gr}_{\mathbf{d}_1, \mathbf{d}_2}^{\max} \mathbb{C}[W \cdot \mathbf{h}])^* \times (\text{gr}_{\mathbf{d}_1, \mathbf{d}_2}^{\max} \mathbb{C}[W \cdot \mathbf{h}]) \rightarrow \mathbb{C}$ for the canonical pairing and using (7.5.6), we deduce

$$\langle\langle b^*(\nabla), a(\Delta_{\mathbf{e}}) \rangle\rangle = \langle\langle a^*(b^*(\nabla)), \Delta_{\mathbf{e}} \rangle\rangle = \nabla_{\mathbf{d}_1, \mathbf{d}_2}(\Delta_{\mathbf{e}})(0) \neq 0.$$

The map a in (7.5.2) is clearly W -equivariant (with respect to the action induced by the W -diagonal action on $\mathbb{C}[\mathfrak{t}] \otimes \mathbb{C}[\mathfrak{t}]$). Thus, we have shown that $a(\Delta_{\mathbf{e}})$ is a nonzero element of the vector space $(\text{gr}_{\mathbf{d}_1, \mathbf{d}_2}^{\max} \mathbb{C}[W \cdot \mathbf{h}])^{\text{sign}}$. In particular, we get $(\text{gr}_{\mathbf{d}_1, \mathbf{d}_2}^{\max} \mathbb{C}[W \cdot \mathbf{h}])^{\text{sign}} \neq 0$.

We know that $\text{gr}^{\max} \mathbb{C}[W \cdot \mathbf{h}]$ is a graded algebra such that $\text{gr}_{i,j}^{\max} \mathbb{C}[W \cdot \mathbf{h}] = 0$ whenever $i > \mathbf{d}_1$ or $j > \mathbf{d}_2$, by Theorem 1.7.2. Also, we have $\dim(\text{gr}_{i,j}^{\max} \mathbb{C}[W \cdot \mathbf{h}])^{\text{sign}} = 1$ and Theorem 1.7.2 says that the W -equivariant projection $\text{gr}^{\max} \mathbb{C}[W \cdot \mathbf{h}] \twoheadrightarrow (\text{gr}^{\max} \mathbb{C}[W \cdot \mathbf{h}])^{\text{sign}}$ gives a nondegenerate trace on the algebra $\text{gr}^{\max} \mathbb{C}[W \cdot \mathbf{h}]$. Therefore, since $(\text{gr}_{\mathbf{d}_1, \mathbf{d}_2}^{\max} \mathbb{C}[W \cdot \mathbf{h}])^{\text{sign}} \neq 0$, it follows that we must have $\text{gr}_{\mathbf{d}_1, \mathbf{d}_2}^{\max} \mathbb{C}[W \cdot \mathbf{h}] = (\text{gr}_{\mathbf{d}_1, \mathbf{d}_2}^{\max} \mathbb{C}[W \cdot \mathbf{h}])^{\text{sign}} = (\text{gr}^{\max} \mathbb{C}[W \cdot \mathbf{h}])^{\text{sign}}$. In particular, the vector space $\text{gr}_{\mathbf{d}_1, \mathbf{d}_2}^{\max} \mathbb{C}[W \cdot \mathbf{h}]$ is 1-dimensional, with $a(\Delta_{\mathbf{e}})$ being a base vector. The theorem follows. \square

7.6. The algebra $\mathcal{R}_{\mathbf{e}}$. We use the notation of the previous section and keep the assumption that \mathbf{e} is a non-exceptional principal nilpotent pair.

Proof of Corollary 1.7.4. Let $\mathcal{O}_{\mathbf{e},0}$ denote the local ring of the algebra $\mathbb{C}[\mathfrak{X}]$ at the point $(\mathbf{e}, 0) \in \mathfrak{G} \times \mathfrak{T}$. The variety \mathfrak{X} is normal at $(\mathbf{e}, 0)$ if and only if the canonical imbedding $\mathcal{O}_{\mathbf{e},0} \rightarrow \mathcal{O}_{\mathbf{e},0} \otimes_{\mathbb{C}[\mathfrak{X}]} \mathbb{C}[\mathfrak{X}_{\text{norm}}]$ is an isomorphism. The latter holds, by the Nakayama lemma, if and only if the natural map $\mathbb{C}[p^{-1}(\mathbf{e})] \rightarrow \mathbb{C}[p_{\text{norm}}^{-1}(\mathbf{e})] = \mathcal{R}_{\mathbf{e}}$ is surjective. This yields the equivalence (i) \Leftrightarrow (ii) of the Corollary, since the restriction map $\mathbb{C}[\mathfrak{T}] \rightarrow \mathbb{C}[p^{-1}(\mathbf{e})]$ is surjective.

The equivalence (ii) \Leftrightarrow (iii) is a direct consequence of Theorem 1.7.2. \square

Proof of Corollary 1.7.3. Let $I = I_{\mathbf{h}}$ be the ideal of the reduced subscheme $W \cdot \mathbf{h} \subset \mathfrak{T}$ and choose $d \gg 0$ such that ${}''F_d \mathbb{C}[W \cdot \mathbf{h}] = \mathbb{C}[W \cdot \mathbf{h}]$ (by Lemma 7.2.1(ii), one can take $d = \mathbf{d}_2$). Thus, we have ${}''F_d \mathbb{C}[\mathfrak{T}] + I = \mathbb{C}[\mathfrak{T}]$. Also, by definition, one has $'F_0 \mathbb{C}[\mathfrak{T}] = 1 \otimes \mathbb{C}[\mathfrak{t}]$. Therefore, we find

$$F_{0,d} \mathbb{C}[W \cdot \mathbf{h}] = \frac{((1 \otimes \mathbb{C}[\mathfrak{t}] + I) \cap ({}''F_d \mathbb{C}[\mathfrak{T}] + I))}{I} = \frac{(1 \otimes \mathbb{C}[\mathfrak{t}] + I)}{I} \cong \frac{1 \otimes \mathbb{C}[\mathfrak{t}]}{(1 \otimes \mathbb{C}[\mathfrak{t}]) \cap I}.$$

The ideal $(1 \otimes \mathbb{C}[t]) \cap I$ corresponds, via the identification $1 \otimes \mathbb{C}[t] = \mathbb{C}[t]$, to the ideal in $\mathbb{C}[t]$ of the image of the orbit $W \cdot \mathbf{h}$ under the second projection $\mathfrak{T} \rightarrow \mathfrak{t}$. This image equals the set $W \cdot h_2 \subset \mathfrak{t}$. We conclude that the rightmost space in the displayed formula above is isomorphic to $\mathbb{C}[W \cdot h_2]$. Thus, using that formula, we obtain W -module isomorphisms

$${}''\text{gr}(F_{0,d}\mathbb{C}[W \cdot \mathbf{h}]) \cong F_{0,d}\mathbb{C}[W \cdot \mathbf{h}] \cong \mathbb{C}[W \cdot h_2] \cong \mathbb{C}[W/W_1].$$

This yields the first isomorphism of the corollary. The second isomorphism is proved similarly. \square

As an immediate consequence of the (proof of) Theorem 1.8.1, we have

Lemma 7.6.1. *There is a nonzero constant $c \in \mathbb{C}$ such that $\Delta_{\mathbf{e}}(w\mathbf{h}) = c \cdot \text{sign}(w)$, $\forall w \in W$.*

Proof. Any W -alternating function on the orbit $W \cdot \mathbf{h}$ is a constant multiple of the function $\text{sign}_{W \cdot \mathbf{h}} : w(\mathbf{h}) \mapsto \text{sign}(w)$. The function $\Delta_{\mathbf{e}}|_{W \cdot \mathbf{h}}$ is W -alternating, hence it must be proportional to the function $\text{sign}_{W \cdot \mathbf{h}}$. We know that the class of $\Delta_{\mathbf{e}}$ in $\text{gr}_{\mathbf{d}_1, \mathbf{d}_2}^{\max} \mathbb{C}[W \cdot \mathbf{h}]$ is nonzero, by Theorem 1.8.1. Hence, the function $\Delta_{\mathbf{e}}|_{W \cdot \mathbf{h}}$ is not identically zero and the result follows. \square

Let f be a holomorphic function on \mathfrak{T} . The pull-back of f via the composite $\mathfrak{X}_{\text{norm}} \rightarrow \mathfrak{X} \rightarrow \mathfrak{T}$ is a G -invariant holomorphic function on $\mathfrak{X}_{\text{norm}}$. One may view that function as a holomorphic section, f^\sharp , of the sheaf \mathcal{R} , resp. view the restriction of that function to the subscheme $p_{\text{norm}}^{-1}(\mathbf{e}) \hookrightarrow \mathfrak{X}_{\text{norm}}$ as an element $\text{res}_{\mathbf{e}}(f) \in \mathcal{R}_{\mathbf{e}} = \mathbb{C}[p_{\text{norm}}^{-1}(\mathbf{e})]$, the value of the section f^\sharp at the point \mathbf{e} .

We know, thanks to Theorem 1.8.1 and Lemma 7.6.1, that the class in $\text{gr}^{\max} \mathbb{C}[W \cdot \mathbf{h}]$ of the function $\text{sign}_{W \cdot \mathbf{h}}$ gives a base vector of the 1-dimensional vector space $(\text{gr}^{\max} \mathbb{C}[W \cdot \mathbf{h}])^{\text{sign}} = \text{gr}_{\mathbf{d}_1, \mathbf{d}_2}^{\max} \mathbb{C}[W \cdot \mathbf{h}]$. Let $u \in \mathcal{R}_{\mathbf{e}}^{\text{sign}}$ be the image of that base vector under the isomorphism $(\text{gr}^{\max} \mathbb{C}[W \cdot \mathbf{h}])^{\text{sign}} \xrightarrow{\sim} \mathcal{R}_{\mathbf{e}}^{\text{sign}}$, of Theorem 1.7.2. Thus, u is a base vector of the 1-dimensional vector space $\mathcal{R}_{\mathbf{e}}^{\text{sign}}$.

It is clear that, for any W -alternating function f , on \mathfrak{T} , there is a constant $c(f) \in \mathbb{C}$ such that one has $\text{res}_{\mathbf{e}}(f) = c(f) \cdot u$.

Proof of Corollary 1.8.4. Let $\text{pr}_{\mathcal{R}}^{\text{sign}} : \mathcal{R}_{\mathbf{e}} \rightarrow \mathcal{R}_{\mathbf{e}}^{\text{sign}}$, resp. $\text{pr}_{W \cdot \mathbf{h}}^{\text{sign}} : \mathbb{C}[W \cdot \mathbf{h}] \rightarrow \mathbb{C}[W \cdot \mathbf{h}]^{\text{sign}}$, be the projection to the sign isotypic component. Further, we will use a natural identification $\zeta : \mathbb{C}[\mathfrak{T}] = \text{gr } \mathbb{C}[\mathfrak{T}]$, as bigraded algebras, resulting from the bigrading on $\mathbb{C}[\mathfrak{T}] = \bigoplus_{i,j} \mathbb{C}^{i,j}[\mathfrak{T}]$ on the algebra $\mathbb{C}[\mathfrak{T}]$ itself. We have a diagram

$$\begin{array}{ccccc} \mathbb{C}[\mathfrak{T}] & \xrightarrow{\text{res}_{\mathbf{e}}} & \mathcal{R}_{\mathbf{e}} & \xrightarrow{\text{pr}_{\mathcal{R}}^{\text{sign}}} & \mathcal{R}_{\mathbf{e}}^{\text{sign}} \\ \zeta \parallel & & \parallel & & \parallel \\ \text{gr } \mathbb{C}[\mathfrak{T}] & \xrightarrow{a} & \text{gr}^{\max} \mathbb{C}[W \cdot \mathbf{h}] & \xrightarrow{\text{pr}_{W \cdot \mathbf{h}}^{\text{sign}}} & \text{gr}_{\mathbf{d}_1, \mathbf{d}_2}^{\max} \mathbb{C}[W \cdot \mathbf{h}]. \end{array} \quad (7.6.2)$$

Theorem 1.7.2 Theorem 1.8.1

Here, the map a is induced by the projection $\mathbb{C}[\mathfrak{T}] \twoheadrightarrow \mathbb{C}[W \cdot \mathbf{h}]$, cf. (7.5.2).

Going through the construction of the isomorphism of Theorem 1.7.2, see esp. formulas (7.2.7) and (7.3.5), shows that the left square in diagram (7.6.2) commutes. The right square commutes trivially.

For any W -alternating function f on \mathfrak{T} we have $f|_{W \cdot \mathbf{h}} = f(\mathbf{h}) \cdot \text{sign}_{W \cdot \mathbf{h}}$. Assume, in addition, that $f \in (\mathbb{C}^{\mathbf{d}_1, \mathbf{d}_2}[\mathfrak{T}])^{\text{sign}}$, a homogeneous polynomial of bidegree $(\mathbf{d}_1, \mathbf{d}_2)$. Then, the equation $f|_{W \cdot \mathbf{h}} = f(\mathbf{h}) \cdot \text{sign}_{W \cdot \mathbf{h}}$, in $\mathbb{C}[W \cdot \mathbf{h}]$, implies an equation $a(\zeta(f)) = f(\mathbf{h}) \cdot \text{sign}_{W \cdot \mathbf{h}}$, in $\text{gr}_{\mathbf{d}_1, \mathbf{d}_2}^{\max} \mathbb{C}[W \cdot \mathbf{h}]$.

We transport the latter equation via the isomorphism of Theorem 1.7.2. Thus, using commutativity of diagram (7.6.2) and our definition of the base vector $u \in \mathcal{R}^{sign}$, we deduce

$$res_e(f) = f(\mathbf{h}) \cdot u, \quad \forall f \in (\mathbb{C}^{\mathbf{d}_1, \mathbf{d}_2}[\mathfrak{T}])^{sign}. \quad (7.6.3)$$

Next, by definition one has $s_e = \Delta_e^\sharp$. Hence, $s_e(e) = res_e(\Delta_e)$. Thus, applying formula (7.6.3) to the polynomial Δ_e , we get $s_e(e) = \Delta_e(\mathbf{h}) \cdot u$.

Now, Lemma 7.6.1 implies that $\Delta_e(\mathbf{h}) \neq 0$. It follows that $s_e(e) \neq 0$. \square

Remark 7.6.4. We have shown in the course of the proof of Theorem 1.8.1 that, we have $\nabla_{\mathbf{d}_1, \mathbf{d}_2}(\Delta_e)(0) \neq 0$. This provides an alternative proof that $\Delta_e(\mathbf{h}) \neq 0$, thanks to a simple formula $\nabla_{\mathbf{d}_1, \mathbf{d}_2}(\Delta_e)(0) = (\mathbf{d}_1! \mathbf{d}_2!)^2 \cdot \Delta_e(\mathbf{h})$.

Proof of Propositions 1.8.5. For each $\mathbf{x} \in \mathfrak{T}$, we let $\mathbf{E}_{\mathbf{x}}(-) := \mathbf{E}(\mathbf{x}, -)$, a W -alternating holomorphic function on \mathfrak{T} . Unraveling definitions, one finds that $\mathbf{S}|_{\mathfrak{C} \times \{\mathbf{e}\}} = s^\sharp \boxtimes u$, where s^\sharp is a G -invariant holomorphic section of the sheaf \mathcal{R}^{sign} that corresponds to the holomorphic function $s : \mathfrak{T} \rightarrow \mathbb{C}$, $\mathbf{x} \mapsto c(\mathbf{E}_{\mathbf{x}})$.

Fix $\mathbf{x} = (x_1, x_2) \in \mathfrak{T}$. For each $i, j \geq 1$, we define a homogeneous polynomial on \mathfrak{T} of bidegree (i, j) by the formula

$$\mathbf{E}_{\mathbf{x}}^{ij}(y_1, y_2) = \frac{1}{i!j!} \sum_{w \in W} sign(w) \cdot (\langle x_1, w(y_1) \rangle)^i (\langle x_2, w(y_2) \rangle)^j.$$

Writing $\mathbf{y} = (y_1, y_2) \in \mathfrak{T}$, one obtains an obvious expansion

$$\mathbf{E}_{\mathbf{x}}(\mathbf{y}) = \sum_{w \in W} sign(w) \cdot e^{\langle \mathbf{x}, w(\mathbf{y}) \rangle} = \sum_{w \in W} sign(w) \cdot e^{\langle x_1, w(y_1) \rangle} \cdot e^{\langle x_2, w(y_2) \rangle} = \sum_{i, j \geq 0} \mathbf{E}_{\mathbf{x}}^{ij}(y_1, y_2).$$

Observe next that the composite map $\text{pr}_{W \cdot \mathbf{h}}^{sign} \circ a$, in the bottom row of diagram (7.6.2), annihilates the homogeneous component $\mathbb{C}^{i, j}[\mathfrak{T}]$ unless we have $(i, j) = (\mathbf{d}_1, \mathbf{d}_2)$. From the commutativity of the diagram we deduce that the map $\text{pr}_{\mathcal{R}}^{sign} \circ res_e$, in the top row of the diagram, annihilates all the components $\mathbb{C}^{i, j}[\mathfrak{T}]$, $(i, j) \neq (\mathbf{d}_1, \mathbf{d}_2)$. It follows that we have $\text{pr}_{\mathcal{R}}^{sign}(res_e(\mathbf{E}_{\mathbf{x}})) = \text{pr}_{\mathcal{R}}^{sign}(res_e(\mathbf{E}_{\mathbf{x}}^{\mathbf{d}_1, \mathbf{d}_2}))$. Since each of the functions $\mathbf{E}_{\mathbf{x}}^{\mathbf{d}_1, \mathbf{d}_2}$ and $\mathbf{E}_{\mathbf{x}}$ is W -alternating, we deduce $res_e(\mathbf{E}_{\mathbf{x}}) = res_e(\mathbf{E}_{\mathbf{x}}^{\mathbf{d}_1, \mathbf{d}_2})$. Hence, applying formula (7.6.3), we obtain

$$res_e(\mathbf{E}_{\mathbf{x}}) = res_e(\mathbf{E}_{\mathbf{x}}^{\mathbf{d}_1, \mathbf{d}_2}) = \mathbf{E}_{\mathbf{x}}^{\mathbf{d}_1, \mathbf{d}_2}(\mathbf{h}) \cdot u,$$

where u is the base vector of \mathcal{R}_e^{sign} defined earlier in this section.

Now, using the definition of the polynomial Δ_e , we compute

$$\mathbf{E}_{\mathbf{x}}^{\mathbf{d}_1, \mathbf{d}_2}(\mathbf{h}) = \frac{1}{\mathbf{d}_1! \mathbf{d}_2!} \sum_{w \in W} sign(w) \cdot (\langle x_1, w(h_1) \rangle)^{\mathbf{d}_1} (\langle x_2, w(h_2) \rangle)^{\mathbf{d}_2} = \frac{1}{\mathbf{d}_1! \mathbf{d}_2!} \cdot \Delta_e(\mathbf{x}).$$

Recall finally the function $s : \mathbf{x} \mapsto c(\mathbf{E}_{\mathbf{x}})$, where the quantity $c(\mathbf{E}_{\mathbf{x}})$ is determined from the equation $res_e(\mathbf{E}_{\mathbf{x}}) = c(\mathbf{E}_{\mathbf{x}}) \cdot u$. Thus, using the above formulas, we find

$$s(\mathbf{x}) = c(\mathbf{E}_{\mathbf{x}}) = \mathbf{E}_{\mathbf{x}}^{\mathbf{d}_1, \mathbf{d}_2}(\mathbf{h}) = \frac{1}{\mathbf{d}_1! \mathbf{d}_2!} \cdot \Delta_e(\mathbf{x}).$$

Since $s_e = \Delta_e^\sharp$, for the corresponding sections of the sheaf \mathcal{R} we deduce an equation $s^\sharp = \frac{1}{\mathbf{d}_1! \mathbf{d}_2!} \cdot s_e$. Also, according to the proof of Corollary 1.8.4, we have $s_e(e) = \Delta_e(\mathbf{h}) \cdot u$.

Combining everything together, we obtain

$$\mathbf{S}|_{\mathfrak{C} \times \{\mathbf{e}\}} = s^\sharp \boxtimes u = \frac{1}{\mathbf{d}_1! \cdot \mathbf{d}_2!} \cdot s_e \boxtimes u = \frac{1}{\mathbf{d}_1! \cdot \mathbf{d}_2! \cdot \Delta_e(\mathbf{h})} \cdot s_e \boxtimes s_e(e),$$

proving the first equation of Proposition 1.8.5. The proof of the second equation is similar. \square

8. RELATION TO WORK OF M. HAIMAN

8.1. A vector bundle on the Hilbert scheme. We keep the notation of §1.9, so $G = GL_n$ and $V = \mathbb{C}^n$. We have $\mathfrak{g} = \text{End}_{\mathbb{C}} V$. For any $(x, y) \in \mathfrak{C}$, one may view the vector space $\mathfrak{g}_{x,y}$ as an associative subalgebra of $\text{End}_{\mathbb{C}} V$. Let $G_{x,y} \subset G$ be the isotropy group of the pair (x, y) under the G -diagonal action. The group $G_{x,y}$ may be identified with the group of invertible elements of the associative algebra $\mathfrak{g}_{x,y}$ hence, this group is connected.

Let $\mathbb{C}\langle x, y \rangle$ denote an associative subalgebra of $\text{End}_{\mathbb{C}} V$ generated by the elements x and y . Recall that a vector $v \in V$ is said to be a *cyclic vector* for a pair $(x, y) \in \mathfrak{C}$ if one has $\mathbb{C}\langle x, y \rangle v = V$. Let \mathfrak{C}° be the set of pairs $(x, y) \in \mathfrak{C}$ which have a cyclic vector.

Part (i) of the following result is due to Neubauer and Saltman and part (ii) is well-known, cf. [NS, Theorem 1.1].

Lemma 8.1.1. (i) *A pair $(x, y) \in \mathfrak{C}$ is regular if and only if one has $\mathbb{C}\langle x, y \rangle = \mathfrak{g}_{x,y}$.*

(ii) *The set \mathfrak{C}° is a Zariski open subset of \mathfrak{C} . For $(x, y) \in \mathfrak{C}^\circ$, all cyclic vectors for (x, y) form a single $G_{x,y}$ -orbit, which is the unique Zariski open dense $G_{x,y}$ -orbit in V .* \square

Let G act on $\mathfrak{C} \times V$ by $g : (x, y, v) \mapsto (gxg^{-1}, gyg^{-1}, gv)$. We introduce a variety of triples

$$\mathcal{S} = \{(x, y, v) \in \mathfrak{g} \times \mathfrak{g} \times V \mid [x, y] = 0, \mathbb{C}\langle x, y \rangle v = V\}. \quad (8.1.2)$$

This is a Zariski open G -stable subset of $\mathfrak{C} \times V$. Lemma 8.1.1 shows that \mathcal{S} is smooth and that the G -action on \mathcal{S} is free. It is known that there exists a universal geometric quotient morphism $\rho : \mathcal{S} \rightarrow \mathcal{S}/G$. Furthermore, the variety \mathcal{S}/G may be identified with $\text{Hilb} := \text{Hilb}^n(\mathbb{C}^2)$, the Hilbert scheme of n points in \mathbb{C}^2 , see [Na].

The first projection $\mathfrak{C} \times V \rightarrow \mathfrak{C}$ restricts to a G -equivariant map $\delta : \mathcal{S} \rightarrow \mathfrak{C}^\circ$. We let $\mathcal{G}_{\mathfrak{C}^\circ}$, the universal stabilizer group scheme on \mathfrak{C}° , cf. §5.4, act along the fibers of δ by $(\gamma, x, y) : (x, y, v) \mapsto (x, y, \gamma v)$. Lemma 8.1.1(i) implies that $\mathcal{G}_{\mathfrak{C}^\circ}$ is a smooth commutative group scheme, furthermore, the $\mathcal{G}_{\mathfrak{C}^\circ}$ -action makes $\mathcal{S} \rightarrow \mathfrak{C}^\circ$ a G -equivariant $\mathcal{G}_{\mathfrak{C}^\circ}$ -torsor over \mathfrak{C}° . In particular, δ is a smooth morphism.

We use the notation of §1.5 and put $\mathcal{P} = (\rho_* \delta^*(\mathcal{R}|_{\mathfrak{C}^\circ}))^G$. There is a W -action on \mathcal{P} inherited from the one on \mathcal{R} . By equivariant descent, one has a canonical isomorphism $\rho^* \mathcal{P} = \delta^*(\mathcal{R}|_{\mathfrak{C}^\circ})$, cf. diagram (8.2.1) below. The sheaf $\mathcal{R}|_{\mathfrak{C}^\circ}$ is locally free (Theorem 1.5.2) and self-dual (Theorem 1.3.4). Thus, we deduce the following result

Corollary 8.1.3. *The sheaf \mathcal{P} is a locally free coherent sheaf of commutative $\mathcal{O}_{\text{Hilb}}$ -algebras equipped with a natural W -action by algebra automorphisms. The fibers of the corresponding algebraic vector bundle on Hilb afford the regular representation of the group W and the projection $\mathcal{P} \rightarrow \mathcal{P}^{\text{sign}}$ induces a nondegenerate trace on each fiber.* \square

8.2. The isospectral Hilbert scheme. Let $\mathfrak{W} := \text{Spec}_{\text{Hilb}} \mathcal{P}$ be the relative spectrum of \mathcal{P} , a sheaf of algebras on the Hilbert scheme. By Corollary 8.1.3, the scheme \mathfrak{W} comes equipped with a flat and finite morphism $\eta : \mathfrak{W} \rightarrow \text{Hilb}$ and with a W -action along the fibers of η . We conclude that \mathfrak{W} is a reduced Cohen-Macaulay and Gorenstein variety.

One can interpret the construction of the scheme \mathfrak{W} in more geometric terms as follows. Let $\mathfrak{X}^\circ := p_{\text{norm}}^{-1}(\mathfrak{C}^\circ)$, an open subvariety in $\mathfrak{X}_{\text{norm}}$. We form a fiber product $\mathfrak{X}_{\text{norm}} \times_{\mathfrak{C}^\circ} \mathcal{S}$ and consider the following commutative diagram where \tilde{p} denotes the second projection, cf. also [Ha2, §8],

$$\begin{array}{ccccccc}
\mathfrak{X}^\circ & \xleftarrow{\tilde{\delta}} & \mathfrak{X}^\circ \times_{\mathfrak{C}^\circ} \mathcal{S} & \xrightarrow[\cong]{h} & \mathcal{S} \times_{\text{Hilb}} \mathfrak{W} & \xrightarrow{\tilde{\rho}} & \mathfrak{W} \\
& \searrow p_{\text{norm}} & \searrow \tilde{p} & & \searrow \tilde{\eta} & \searrow \eta & \\
& & \mathfrak{C}^\circ & \xleftarrow{\delta} & \mathcal{S} & \xrightarrow{\rho} & \text{Hilb} \\
& & \searrow (1.3.1) & & \searrow \text{Hilbert-Chow} & & \\
& & & & \mathfrak{T}/W & &
\end{array} \tag{8.2.1}$$

We let G act on $\mathcal{S} \times_{\text{Hilb}} \mathfrak{W}$ through the first factor, resp. act *diagonally* on $\mathfrak{X}^\circ \times_{\mathfrak{C}^\circ} \mathcal{S}$. The morphisms δ and ρ in diagram (8.2.1) are smooth, resp. the morphisms p_{norm} and η are finite and flat. The morphism $\tilde{\delta}$, resp. $\tilde{\rho}$, \tilde{p} , and $\tilde{\eta}$, is obtained from δ , resp. from ρ , p_{norm} , and η , by base change. Hence, flat base change yields

$$\delta^*(\mathcal{R}|_{\mathfrak{C}^\circ}) = \delta^*(p_{\text{norm}})_* \mathcal{O}_{\mathfrak{X}^\circ} = \tilde{p}_* \mathcal{O}_{\mathfrak{X}^\circ \times_{\mathfrak{C}^\circ} \mathcal{S}}, \quad \text{resp.} \quad \rho^* \mathcal{P} = \rho^* \eta_* \mathcal{O}_{\mathfrak{W}} = \tilde{\eta}_* \mathcal{O}_{\mathcal{S} \times_{\text{Hilb}} \mathfrak{W}}. \tag{8.2.2}$$

Since $\delta^*(\mathcal{R}|_{\mathfrak{C}^\circ}) = \rho^* \mathcal{P}$ we deduce a canonical isomorphism $\tilde{p}_* \mathcal{O}_{\mathfrak{X}^\circ \times_{\mathfrak{C}^\circ} \mathcal{S}} \cong \tilde{\eta}_* \mathcal{O}_{\mathcal{S} \times_{\text{Hilb}} \mathfrak{W}}$ of G -equivariant sheaves of $\mathcal{O}_{\mathcal{S}}$ -algebras. This means that there is a canonical G -equivariant isomorphism of schemes $h : \mathfrak{X}^\circ \times_{\mathfrak{C}^\circ} \mathcal{S} \xrightarrow{\sim} \mathcal{S} \times_{\text{Hilb}} \mathfrak{W}$, the dotted arrow in diagram (8.2.1).

The map $\tilde{\rho}$ makes $\mathcal{S} \times_{\text{Hilb}} \mathfrak{W}$ a G -torsor over \mathfrak{W} . Therefore, the composite $\tilde{\rho} \circ h$ makes $\mathfrak{X}^\circ \times_{\mathfrak{C}^\circ} \mathcal{S}$ a G -torsor over \mathfrak{W} hence, we have $\mathfrak{W} = (\mathfrak{X}^\circ \times_{\mathfrak{C}^\circ} \mathcal{S})/G$, a geometric quotient by G . From (8.2.2), we obtain $\rho_* \delta^*(\mathcal{R}|_{\mathfrak{C}^\circ}) = \rho_* \tilde{p}_* \mathcal{O}_{\mathfrak{X}^\circ \times_{\mathfrak{C}^\circ} \mathcal{S}} = \rho_* \tilde{\eta}_* \mathcal{O}_{\mathcal{S} \times_{\text{Hilb}} \mathfrak{W}} = \eta_* \tilde{\rho}_* \mathcal{O}_{\mathcal{S} \times_{\text{Hilb}} \mathfrak{W}}$. Therefore, taking G -invariants, we get $\mathcal{P} = [\rho_* \delta^*(\mathcal{R}|_{\mathfrak{C}^\circ})]^G = [\eta_* \tilde{\rho}_* \mathcal{O}_{\mathcal{S} \times_{\text{Hilb}} \mathfrak{W}}]^G = \eta_* \mathcal{O}_{\mathfrak{W}}$.

The W -action on \mathfrak{X}° induces one on $\mathfrak{X}^\circ \times_{\mathfrak{C}^\circ} \mathcal{S}$. This W -action commutes with the G -diagonal action, hence, descends to $(\mathfrak{X}^\circ \times_{\mathfrak{C}^\circ} \mathcal{S})/G$. The resulting W -action may be identified with the one that comes from the W -action on the sheaf \mathcal{P} , see Corollary 8.1.3. The composite map $\mathfrak{X}^\circ \times_{\mathfrak{C}^\circ} \mathcal{S} \rightarrow \mathfrak{X}^\circ \rightarrow [\mathfrak{C}^\circ \times_{\mathfrak{C}^\circ // G} \mathfrak{T}]_{\text{red}} \rightarrow \mathfrak{T}$ descends to a W -equivariant map $\sigma : \mathfrak{W} \rightarrow \mathfrak{T}$. Thus, we have a diagram

$$\mathfrak{X}^\circ \xleftarrow{\tilde{\delta}} \mathfrak{X}^\circ \times_{\mathfrak{C}^\circ} \mathcal{S} \xrightarrow{\tilde{\rho} \circ h} (\mathfrak{X}^\circ \times_{\mathfrak{C}^\circ} \mathcal{S})/G = \mathfrak{W} \xrightarrow{\eta \times \sigma} \text{Hilb} \times_{\mathfrak{T}/W} \mathfrak{T}. \tag{8.2.3}$$

Following Haiman, one defines the *isospectral Hilbert scheme* to be $[\text{Hilb} \times_{\mathfrak{T}/W} \mathfrak{T}]_{\text{red}}$, a reduced fiber product.

Proposition 8.2.4. *The map $\eta \times \sigma$ on the right of (8.2.3) factors through an isomorphism*

$$\mathfrak{W} \xrightarrow{\sim} [\text{Hilb} \times_{\mathfrak{T}/W} \mathfrak{T}]_{\text{norm}}.$$

In particular, the normalization of the isospectral Hilbert scheme is Cohen-Macaulay and Gorenstein.

Proof. It is clear that $\tilde{\rho} \circ h(\tilde{\delta}^{-1}(\mathfrak{X}^{rs}))$ is a Zariski open subset of \mathfrak{W} . Lemma 2.1.3(ii) implies readily that the map $\eta \times \sigma$ restricts to an isomorphism $\tilde{\rho} \circ h(\tilde{\delta}^{-1}(\mathfrak{X}^{rs})) \xrightarrow{\sim} \text{Hilb} \times_{\mathfrak{T}/W} \mathfrak{T}^r$; in particular, $\eta \times \sigma$ is a birational isomorphism. The image of the map $\eta \times \sigma$ is contained in $[\text{Hilb} \times_{\mathfrak{T}/W} \mathfrak{T}]_{\text{red}}$ since the scheme \mathfrak{W} is reduced.

Further, Corollary 8.1.3 and Lemma 2.6.4 imply that the scheme \mathfrak{W} is Cohen-Macaulay and smooth in codimension 1. We conclude that \mathfrak{W} is a normal scheme which is birational and finite over $[\text{Hilb} \times_{\mathfrak{T}/W} \mathfrak{T}]_{\text{red}}$. This yields the isomorphism of the proposition. \square

As a consequence of Haiman's work, since $\mathcal{P} = \eta_* \mathcal{O}_{\mathfrak{W}}$ we deduce

Corollary 8.2.5. *The vector bundle \mathcal{P} on Hilb is isomorphic to the Procesi bundle, cf. [Ha1]. \square*

Remark 8.2.6. For any $(x, y) \in \mathfrak{C}$, one has a surjective evaluation homomorphism $\mathbb{C}[z_1, z_2] \twoheadrightarrow \mathbb{C}\langle x, y \rangle$, $P \mapsto P(x, y)$. The kernel of this homomorphism is an ideal $I_{x,y} \subset \mathbb{C}[z_1, z_2]$. If $(x, y) \in \mathfrak{C}^r$ then we have $\mathbb{C}[z_1, z_2]/I_{x,y} = \mathfrak{g}_{x,y}$, by Lemma 8.1.1(i). Hence, $I_{x,y}$ has codimension n in $\mathbb{C}[z_1, z_2]$. Therefore, the assignment $(x, y) \mapsto I_{x,y}$ gives a well defined morphism $\varrho : \mathfrak{C}^r \twoheadrightarrow \text{Hilb}$ such that one has $(\varrho|_{\mathfrak{C}^\circ}) \circ \delta = \rho$. It follows that $\varrho^*\mathcal{P}$, the pull back of the sheaf \mathcal{P} , is a $G \times W$ -equivariant locally free sheaf of $\mathcal{O}_{\mathfrak{C}^r}$ -algebras. Furthermore, we have canonical isomorphisms $\delta^*(\varrho^*\mathcal{P}) = \rho^*\mathcal{P} = \delta^*\mathcal{R}$ of $G \times W \times \mathcal{G}_{\mathfrak{C}^\circ}$ -equivariant sheaves of $\mathcal{O}_{\mathcal{S}}$ -algebras. By equivariant descent, this yields a canonical isomorphism $(\varrho^*\mathcal{P})|_{\mathfrak{C}^\circ} \cong \mathcal{R}|_{\mathfrak{C}^\circ}$. Since $\mathfrak{C}^r \setminus \mathfrak{C}^\circ$ is a codimension ≥ 2 subset in the smooth variety \mathfrak{C}^r and the sheaves $\varrho^*\mathcal{P}$ and $\mathcal{R}|_{\mathfrak{C}^r}$ are both locally free, we deduce that the above isomorphism extends canonically to an isomorphism $\varrho^*\mathcal{P} \cong \mathcal{R}|_{\mathfrak{C}^r}$ of $G \times W$ -equivariant locally free sheaves of $\mathcal{O}_{\mathfrak{C}^r}$ -algebras. In other words, there is a $G \times W$ -equivariant isomorphism $\mathfrak{C}^r \times_{\text{Hilb}} \mathfrak{W} \cong p_{\text{norm}}^{-1}(\mathfrak{C}^r)$ of schemes over \mathfrak{C}^r .

8.3. Proof of Theorem 1.9.1. Given a G -variety X and a rational representation E of G we put $\text{Map}_G(X, E) = (\mathbb{C}[X] \otimes E)^G$, the vector space of G -equivariant polynomial maps $X \rightarrow E$. If $X \rightarrow Y$ is a G -torsor, we let $\mathcal{L}_Y(E)$ be the sheaf of sections of an associated vector bundle $X \times_G E \rightarrow Y$ on Y . By definition, one has $\Gamma(Y, \mathcal{L}_Y(E)) = \text{Map}_G(X, E)$.

We may apply the above to the geometric quotient morphism $\rho : \mathcal{S} \rightarrow \text{Hilb}$. The tautological bundle on the Hilbert scheme is defined to be $\mathcal{V} := \mathcal{L}_{\text{Hilb}}(V)$, a rank n vector bundle associated with the vector representation of $G = GL(V)$ in V .

Remark 8.3.1. According to Lemma 8.1.1, for any $(x, y, v) \in \mathcal{S}$, one has a canonical vector space isomorphism $\mathfrak{g}_{x,y} \rightarrow V$, $a \mapsto a(v)$. This gives a canonical isomorphism $\delta^*(\mathfrak{g}_{\mathfrak{C}^\circ}) \xrightarrow{\sim} \rho^*\mathcal{V}$, of G -equivariant sheaves on \mathcal{S} . \diamond

For any $m \geq 0$, one has a vector bundle $\mathcal{L}_{\mathfrak{W}}(V^{\otimes m})$ on \mathfrak{W} associated with the G -representation $V^{\otimes m}$ and with the G -torsor $\tilde{\rho} \circ h : \mathfrak{X}^\circ \times_{\mathfrak{C}^\circ} \mathcal{S} \rightarrow \mathfrak{W}$. Clearly, we have $\mathcal{L}_{\mathfrak{W}}(V^{\otimes m}) = \eta^*(\mathcal{L}_{\text{Hilb}}(V^{\otimes m})) = \eta^*\mathcal{V}^{\otimes m}$ where $\eta : \mathfrak{W} \rightarrow \text{Hilb}$ is the map from (8.2.1). Hence, the projection formula yields

$$\text{Map}_G(\mathfrak{X}^\circ \times_{\mathfrak{C}^\circ} \mathcal{S}, V^{\otimes m}) = \Gamma(\mathfrak{W}, \mathcal{L}_{\mathfrak{W}}(V^{\otimes m})) = \Gamma(\text{Hilb}, \eta_*\eta^*\mathcal{V}^{\otimes m}) = \Gamma(\text{Hilb}, \mathcal{P} \otimes \mathcal{V}^{\otimes m}),$$

where we have used that $\mathcal{P} = \eta_*\mathcal{O}_{\mathfrak{W}}$. Let Φ_1 be the composite of the above isomorphisms.

Write $u \mapsto (x_u, y_u)$, $x_u, y_u \in \mathfrak{g}$ for the projection $\mathfrak{X}_{\text{norm}} \rightarrow \mathfrak{C}_{\text{red}}$. Then, by definition, we have $\mathfrak{X}^\circ \times_{\mathfrak{C}^\circ} \mathcal{S} = \{(u, v), u \in \mathfrak{X}_{\text{norm}}, v \in V \mid \mathbb{C}[x_u, y_u]v = V\}$. Thus, there is a natural open imbedding $\alpha : \mathfrak{X}^\circ \times_{\mathfrak{C}^\circ} \mathcal{S} \hookrightarrow \mathfrak{X}_{\text{norm}} \times V$ and one has the corresponding restriction morphism

$$\alpha^* : \text{Map}_G(\mathfrak{X}_{\text{norm}} \times V, V^{\otimes m}) \longrightarrow \text{Map}_G(\mathfrak{X}^\circ \times_{\mathfrak{C}^\circ} \mathcal{S}, V^{\otimes m}). \quad (8.3.2)$$

Let $\mathbb{C}^\times \subset GL_n$ be the group of scalar matrices. The scalar matrix $z \cdot \text{Id} \in GL_n$, $z \in \mathbb{C}^\times$, acts trivially on $\mathfrak{C}_{\text{red}}$ hence, for any $\ell \geq 0$, the element $z \cdot \text{Id}$ acts on the subspace $\mathbb{C}[\mathfrak{C}_{\text{red}}] \otimes \mathbb{C}^\ell[V] \subset \mathbb{C}[\mathfrak{C}_{\text{red}} \times V]$ via multiplication by $z^{-\ell}$. It follows that the natural inclusion $\mathbb{C}^m[V] \hookrightarrow \mathbb{C}[V]$ induces an isomorphism $\mathbb{C}^m[V] \otimes V^{\otimes m} \xrightarrow{\sim} (\mathbb{C}[V] \otimes V^{\otimes m})^{\mathbb{C}^\times}$. The group $G = GL_n$ being generated by the subgroups \mathbb{C}^\times and SL_n , we obtain a chain of isomorphisms

$$(\mathbb{C}[\mathfrak{X}_{\text{norm}}] \otimes \mathbb{C}^m[V] \otimes V^{\otimes m})^{SL_n} \xrightarrow{\sim} \text{Map}_G(\mathfrak{X}_{\text{norm}}, \mathbb{C}[V] \otimes V^{\otimes m}) \xrightarrow{\sim} \text{Map}_G(\mathfrak{X}_{\text{norm}} \times V, V^{\otimes m}).$$

Write Φ_2 for the composite isomorphism.

The isomorphism of Theorem 1.9.1(i) is defined as the following composition

$$\begin{aligned} (\mathbb{C}[\mathfrak{X}_{\text{norm}}] \otimes \mathbb{C}^m[V] \otimes V^{\otimes m})^{SL_n} &\xrightarrow[\sim]{\Phi_2} \text{Map}_G(\mathfrak{X}_{\text{norm}} \times V, V^{\otimes m}) \\ &\xrightarrow{\alpha^*} \text{Map}_G(\mathfrak{X}^\circ \times_{\mathfrak{C}^\circ} \mathcal{S}, V^{\otimes m}) \xrightarrow[\sim]{\Phi_1} \Gamma(\text{Hilb}, \mathcal{P} \otimes V^{\otimes m}). \end{aligned}$$

All the above maps are clearly $W \times S_m$ -equivariant bigraded $\mathbb{C}[\mathfrak{T}]$ -module morphisms. We see that proving the theorem reduces to the following result

Lemma 8.3.3. *The restriction map α^* in (8.3.2) is an isomorphism.*

Proof. The map α^* is injective since the set $\mathfrak{X}^\circ \times_{\mathfrak{C}^\circ} \mathcal{S}$ is Zariski dense in $\mathfrak{X}_{\text{norm}} \times V$.

To prove surjectivity of α^* recall that a linear operator $x : V \rightarrow V$ has a cyclic vector if and only if $x \in \mathfrak{g}^r$. Also, for any $(x_1, x_2) \in \mathfrak{C}$ and $v \in V$, either of the two equations, $\mathbb{C}[x_1]v = V$ or $\mathbb{C}[x_2]v = V$, implies $\mathbb{C}[x_1, x_2]v = V$. It follows that we have $\mathfrak{C}^{rr} = \mathfrak{C}_1 \cup \mathfrak{C}_2 \subset \mathfrak{C}^\circ$ where \mathfrak{C}_i , $i = 1, 2$, are the open sets introduced in Definition 2.6.3.

Next, let \mathcal{S}_i , $i = 1, 2$, be the open subset of \mathcal{S} formed by the triples $(x_1, x_2, v) \in \mathfrak{C} \times V$ such that we have $\mathbb{C}[x_i]v = V$. Restricting the map $\delta : \mathcal{S} \rightarrow \mathfrak{C}^\circ$, $(x_1, x_2, v) \mapsto (x_1, x_2)$ gives well defined maps $\mathcal{S}_i \rightarrow \mathfrak{C}_i$. This way, we get a commutative diagram of open imbeddings

$$\begin{array}{ccccc} \mathfrak{X}_i \times_{\mathfrak{C}_i} \mathcal{S}_i & \hookrightarrow & \mathfrak{X}^{rr} \times_{\mathfrak{C}^{rr}} \mathcal{S} & \xrightarrow{\beta_{\mathcal{S}}} & \mathfrak{X}^\circ \times_{\mathfrak{C}^\circ} \mathcal{S} \\ \downarrow \alpha_i & & \downarrow \alpha^{rr} & & \downarrow \alpha \\ (\mathfrak{X}_i \times V) & \hookrightarrow & \mathfrak{X}^{rr} \times V & \xrightarrow{\beta_V} & \mathfrak{X}_{\text{norm}} \times V \end{array} \quad (8.3.4)$$

Recall that the morphism $\delta : \mathcal{S} \rightarrow \mathfrak{C}^\circ$ is smooth and the set $\mathfrak{X}_{\text{norm}} \setminus \mathfrak{X}^{rr}$ has codimension ≥ 2 in $\mathfrak{X}_{\text{norm}}$, by Lemma 2.6.4. Hence, the complement of the set $\mathfrak{X}^{rr} \times_{\mathfrak{C}^{rr}} \mathcal{S}$ has codimension ≥ 2 in $\mathfrak{X}^\circ \times_{\mathfrak{C}^\circ} \mathcal{S}$, resp. the complement of the set $\mathfrak{X}^{rr} \times V$ has codimension ≥ 2 in $\mathfrak{X}_{\text{norm}} \times V$. The varieties $\mathfrak{X}^\circ \times_{\mathfrak{C}^\circ} \mathcal{S}$ and $\mathfrak{X}_{\text{norm}} \times V$ are normal. Therefore, restriction of functions yields the following bijections:

$$(\beta_{\mathcal{S}})^* : \mathbb{C}[\mathfrak{X}^\circ \times_{\mathfrak{C}^\circ} \mathcal{S}] \xrightarrow{\sim} \mathbb{C}[\mathfrak{X}^{rr} \times_{\mathfrak{C}^{rr}} \mathcal{S}], \quad \text{resp. } (\beta_V)^* : \mathbb{C}[\mathfrak{X}_{\text{norm}} \times V] \xrightarrow{\sim} \mathbb{C}[\mathfrak{X}^{rr} \times V]. \quad (8.3.5)$$

Similarly, the inclusion α_i in diagram (8.3.4) induces a restriction map

$$\alpha_i^* : \text{Map}_G(\mathfrak{X}_i \times V, V^{\otimes m}) \rightarrow \text{Map}_G(\mathfrak{X}_i \times_{\mathfrak{C}_i} \mathcal{S}_i, V^{\otimes m}), \quad i = 1, 2. \quad (8.3.6)$$

We will show in Lemma 8.4.1 below that this map is surjective.

To complete the proof of Lemma 8.3.3, let $f : \mathfrak{X}^{rr} \times_{\mathfrak{C}^{rr}} \mathcal{S} \rightarrow V^{\otimes m}$ be a G -equivariant map. Then, thanks to the surjectivity of (8.3.6), there exist maps $f_i \in \text{Map}_G(\mathfrak{X}_i \times V, V^{\otimes m})$, $i = 1, 2$, such that one has $f|_{\mathfrak{X}_i \times_{\mathfrak{C}_i} \mathcal{S}_i} = f_i|_{\mathfrak{X}_i \times_{\mathfrak{C}_i} \mathcal{S}_i}$. It follows that the restrictions of the maps f_1 and f_2 to the set $(\mathfrak{X}_1 \times V) \cap (\mathfrak{X}_2 \times V)$ agree. Therefore, the map f can be extended to a regular map $\mathfrak{X}^{rr} \times V = (\mathfrak{X}_1 \times V) \cup (\mathfrak{X}_2 \times V) \rightarrow V^{\otimes m}$. Thus, we have proved (modulo Lemma 8.4.1) the surjectivity of the restriction map $(\alpha^{rr})^*$ induced by the vertical imbedding α^{rr} in the middle of diagram (8.3.4). This, combined with isomorphisms (8.3.5), implies the surjectivity of the map α^* and Lemma 8.3.3 follows. \square

8.4. To complete the proof of the theorem, it remains to prove the following

Lemma 8.4.1. *The map α_i^* , $i = 1, 2$, in (8.3.6) is surjective.*

Proof. We may restrict ourselves to the case $i = 1$, the case $i = 2$ being similar.

Let $\mathfrak{x} \subset \mathfrak{g} \times \mathfrak{t}$ be the closed subvariety considered in §2.1, and let $\mathfrak{r} = \{(x, t) \in \mathfrak{x} \mid x \in \mathfrak{g}^r\}$. Write $q : \mathfrak{X} \rightarrow \mathfrak{x}$, $(x, y, t_1, t_2) \mapsto (x, t_1)$ for the projection. According to Lemma 2.1.5 we

have that $q^{-1}(\mathfrak{r}) = N_{\mathfrak{r}}$, the total space of the conormal bundle on \mathfrak{r} . Thus, we obtain $\mathfrak{X}_1 = \{(x, y, t_1, t_2) \in \mathfrak{X}, x \in \mathfrak{g}^r\} = q^{-1}(\mathfrak{r}) = N_{\mathfrak{r}}$.

It will be convenient to introduce the following set

$$Y := \{(x, t, v) \in \mathfrak{g} \times \mathfrak{t} \times V \mid (x, t) \in \mathfrak{r} \text{ \& \; } \mathbb{C}[x]v = V\}.$$

Note that, for any $(x, t, v) \in Y$, the element x is automatically regular, as has been already observed earlier. Therefore, we have $Y \subset \mathfrak{r} \times V$ and the projection $(x, t, v) \mapsto (x, t)$ gives a smooth and surjective morphism $Y \rightarrow \mathfrak{r}$. Unraveling the definitions we obtain

$$\mathfrak{X}_1 \times_{\mathfrak{c}_1} \mathcal{S}_1 = \{(x_1, x_2, t_1, t_2, v) \in \mathfrak{X} \times V \mid \mathbb{C}[x_1]v = V\} = q^{-1}(\mathfrak{r}) \times_{\mathfrak{r}} Y = N_{\mathfrak{r}} \times_{\mathfrak{r}} Y.$$

Using these identifications, we see that the map α_1 from (8.3.4) fits into a cartesian square

$$\begin{array}{ccc} N_{\mathfrak{r}} \times_{\mathfrak{r}} Y & \xrightarrow{\alpha_1} & N_{\mathfrak{r}} \times V \\ \downarrow pr_Y & & \downarrow q_V := q \times \text{Id}_V \\ Y & \hookrightarrow & \mathfrak{r} \times V \end{array} \quad (8.4.2)$$

Let $\mathbf{D} := (\mathfrak{r} \times V) \setminus Y$. This is an irreducible divisor in $\mathfrak{r} \times V$, the principal divisor associated with the function $(x, t, v) \mapsto \langle \text{vol}, v \wedge x(v) \wedge x^2(v) \wedge \dots \wedge x^{n-1}(v) \rangle$ where $\text{vol} \in \wedge^n V^*$ is a fixed nonzero element. Using the cartesian square in (8.4.2), we see that $(N_{\mathfrak{r}} \times V) \setminus (N_{\mathfrak{r}} \times_{\mathfrak{r}} Y) = (q_V)^{-1}(\mathbf{D})$ is an irreducible divisor in $N_{\mathfrak{r}} \times V$. Now, view \mathbf{D} as a subset of $\mathfrak{g} \times \mathfrak{t} \times V$. Then, $\mathbf{D}^{rs} := \mathbf{D} \cap (\mathfrak{g}^{rs} \times \mathfrak{t} \times V)$ is an open dense subset of \mathbf{D} hence $(q_V)^{-1}(\mathbf{D}^{rs})$ is an open dense subset of $(q_V)^{-1}(\mathbf{D})$.

Let T be the maximal torus of diagonal matrices in GL_n . Thus, the set $\mathfrak{t} \cap \mathfrak{g}^r$ may be identified with the set of matrices $t = \text{diag}(z_1, \dots, z_n) \in \mathbb{C}^n$ such that $z_i \neq z_j \forall i \neq j$. Let $x = t = \text{diag}(z_1, \dots, z_n) \in \mathfrak{t} \cap \mathfrak{g}^r$. For $v = (v_1, \dots, v_n) \in \mathbb{C}^n = V$, the triple (x, t, v) is contained in \mathbf{D}^{rs} if and only if v is not a cyclic vector for the linear operator x which holds if and only if there exists an $i \in [1, n]$ such that $v_i = 0$. Let $V_i \subset V$ denote the hyperplane formed by the elements with the vanishing i -th coordinate.

To complete the proof of the lemma, let $f \in \text{Map}_G(\mathfrak{X}^\circ \times_{\mathfrak{c}^\circ} \mathcal{S}, V^{\otimes m})$. We must show that $(\alpha_1)^*(f)$, viewed as a map $N_{\mathfrak{r}} \times_{\mathfrak{r}} Y \rightarrow V^{\otimes m}$, has no singularities at $(N_{\mathfrak{r}} \times V) \setminus (N_{\mathfrak{r}} \times_{\mathfrak{r}} Y)$, a divisor in a smooth variety. The set $(q_V)^{-1}(\mathbf{D}^{rs})$ is an open dense subset of that divisor. Hence, by G -equivariance, it suffices to prove that, for $x = t \in \mathfrak{t} \cap \mathfrak{g}^{rs}$ and a fixed element $(x, y, t, t') \in q^{-1}(x, t) = N_{\mathfrak{r}}$, the rational map $f_V : V \rightarrow V^{\otimes m}$ given by the assignment $v \mapsto f(x, y, t, t', v)$ has no poles at the divisor $\cup_i V_i \subset V$.

It will be convenient to choose a T -weight basis $\{u_\gamma\}$ of the vector space $V^{\otimes m}$. Thus, for each γ and any diagonal matrix $\text{diag}(z_1, \dots, z_n) \in T$, we have $\text{diag}(z_1, \dots, z_n)u_\gamma = z_1^{m(\gamma,1)} \dots z_n^{m(\gamma,n)} \cdot u_\gamma$ where $m(\gamma, i) \in \mathbb{Z}_{\geq 0}$. Expanding the function f_V in the basis $\{u_\gamma\}$ one can write $f_V(v) = \sum_\gamma f_V^\gamma(v) \cdot u_\gamma$ where f_V^γ are Laurent polynomials of the form

$$v = (v_1, \dots, v_n) \mapsto f_V^\gamma(v) = \sum_{k_1, \dots, k_n \in \mathbb{Z}} a_{k_1, \dots, k_n}^\gamma \cdot v_1^{k_1} \dots v_n^{k_n}, \quad a_{k_1, \dots, k_n}^\gamma \in \mathbb{C}.$$

Now, the map f hence also the map $(\alpha_1)^*(f)$ is G -equivariant. Since $(x, y) \in \mathfrak{T}$ we deduce that f_V is a T -equivariant map. Thus, for each γ and all $\text{diag}(z_1, \dots, z_n) \in T$, we must have

$$\begin{aligned} \sum_{k_1, \dots, k_n \in \mathbb{Z}} a_{k_1, \dots, k_n}^\gamma \cdot (z_1 v_1)^{k_1} \dots (z_n v_n)^{k_n} \cdot u_\gamma &= f_V^\gamma[\text{diag}(z_1, \dots, z_n)v] \\ &= \text{diag}(z_1, \dots, z_n)[f_V^\gamma(v)] = \sum_{k_1, \dots, k_n \in \mathbb{Z}} a_{k_1, \dots, k_n}^\gamma \cdot v_1^{k_1} \dots v_n^{k_n} \cdot [z_1^{m(\gamma,1)} \dots z_n^{m(\gamma,n)} \cdot u_\gamma]. \end{aligned}$$

The above equation yields $k_i = m(\gamma, i)$ whenever $a_{k_1, \dots, k_n}^\gamma \neq 0$. Therefore, the function f_V takes the following simplified form

$$f_V(v_1, \dots, v_n) = \sum_{\gamma} a^\gamma \cdot v_1^{m(\gamma, 1)} \dots v_n^{m(\gamma, n)} \cdot u_\gamma, \quad a^\gamma \in \mathbb{C}. \quad (8.4.3)$$

Since the weights of the T -action in $V^{\otimes m}$ are nonnegative, i.e., $m(\gamma, i) \geq 0$, the right hand side of (8.4.3) has no poles at the divisor $\cup_i V_i$ and the lemma follows. \square

8.5. Proof of Proposition 1.9.2. Let $v_o = (1, 1, \dots, 1) \in \mathbb{C}^n = V$. Restriction of polynomial functions via the imbedding $\zeta : \mathfrak{T} = \mathfrak{T} \times \{v_o\} \hookrightarrow \mathfrak{C}_{\text{red}} \times V$ gives an algebra homomorphism

$$\zeta^* : \mathbb{C}[\mathfrak{C}_{\text{red}}] \otimes \mathbb{C}[V] = \mathbb{C}[\mathfrak{C}_{\text{red}} \times V] \longrightarrow \mathbb{C}[\mathfrak{T}], \quad f \mapsto f|_{\mathfrak{T} \times \{v_o\}}. \quad (8.5.1)$$

Given a G -variety X and an integer $k \geq 0$, let $\mathbb{C}[X]^{\det^k} = \{f \in \mathbb{C}[X] \mid g^*(f) = (\det g)^k \cdot f, \forall g \in G\}$ be the subspace of \det^k -semi-invariants of the group $G = GL_n$. Looking at the action of the group $\mathbb{C}^\times \subset GL_n$, of scalar matrices, shows that one has $\mathbb{C}[\mathfrak{C}_{\text{red}} \times V]^{\det^k} = (\mathbb{C}[\mathfrak{C}_{\text{red}}] \otimes \mathbb{C}^{kn}[V])^{SL_n}$.

Following [GG], we introduce an affine variety

$$\bar{\mathcal{S}} := \{(x, y, v, v^*) \in \mathfrak{g} \times \mathfrak{g} \times V \times V^* \mid [x, y] + v \otimes v^* = 0\}. \quad (8.5.2)$$

The assignment $(x, y, v) \mapsto (x, y, v, 0)$ gives a G -equivariant closed imbedding $\iota : \mathfrak{C}_{\text{red}} \times V \hookrightarrow \bar{\mathcal{S}}$. Pull-back of functions via the imbedding ζ , resp. ι , yields linear maps

$$\mathbb{C}[\bar{\mathcal{S}}]^{\det^k} \xrightarrow{\iota^*} \mathbb{C}[\mathfrak{C}_{\text{red}} \times V]^{\det^k} = (\mathbb{C}[\mathfrak{C}_{\text{red}}] \otimes \mathbb{C}^{kn}[V])^{SL_n} \xrightarrow{\zeta^*} \mathbb{C}^{kn}[\mathfrak{T}], \quad k \geq 0. \quad (8.5.3)$$

The composite map in (8.5.3) was considered in [GG]. According to Proposition A2 from [GG, Appendix], the image of the map $\zeta^* \circ \iota^*$ is equal to A^k and, moreover, this map yields an isomorphism $\mathbb{C}[\bar{\mathcal{S}}]^{\det^k} \xrightarrow{\sim} A^k$. Note further that the restriction map ι^* in (8.5.3) is surjective since the group G is reductive. It follows that each of the two maps in (8.5.3) must be an isomorphism. This yields the statement of the proposition. \square

9. SOME APPLICATIONS

9.1. For each $\text{Ad } G$ -orbit $O \subset \mathfrak{g}$, the closure $\overline{N_O}$ of the total space of the conormal bundle on O is a Lagrangian subvariety in $T^*\mathfrak{g}$. We define

$$\mathfrak{C}^{\text{nil}} := \{(x, y) \in \mathfrak{G} \mid [x, y] = 0, x \in \mathcal{N}\} = \cup_{O \subset \mathcal{N}} \overline{N_O}, \quad (9.1.1)$$

where the union on the right is taken over the (finite) set of nilpotent $\text{Ad } G$ -orbits in \mathfrak{g} .

Let $\gamma : \mathfrak{g} \rightarrow \mathfrak{g} // G = \mathfrak{t} / W$ be the adjoint quotient morphism. We introduce the following maps

$$\theta_{\mathfrak{C}} : \mathfrak{C}_{\text{red}} \rightarrow \mathfrak{g} // G = \mathfrak{t} / W, \quad (x, y) \mapsto \gamma(y) \quad \text{resp.} \quad \theta_{\mathfrak{X}} : \mathfrak{X} \rightarrow \mathfrak{t}, \quad (x, y, t_1, t_2) \mapsto t_2. \quad (9.1.2)$$

The proposition below may be viewed as an analogue of the crucial flatness result in Haiman's proof of the $n!$ theorem, see Proposition 3.8.1 and Corollary 3.8.2 in [Ha1]. Proposition 9.1.3 was also obtained independently by I. Gordon [Go] who used it in his proof of positivity of the Kostka-Macdonald polynomials.

Proposition 9.1.3. *The composite morphism $\tilde{\theta}_{\mathfrak{C}} : \mathfrak{C}_{\text{norm}} \rightarrow \mathfrak{C}_{\text{red}} \xrightarrow{\theta_{\mathfrak{C}}} \mathfrak{g} // G$, resp. $\tilde{\theta}_{\mathfrak{X}} : \mathfrak{X}_{\text{norm}} \rightarrow \mathfrak{X} \xrightarrow{\theta_{\mathfrak{X}}} \mathfrak{t}$ is flat. Furthermore, the scheme theoretic zero fiber $\tilde{\theta}_{\mathfrak{C}}^{-1}(0)$, resp. $\tilde{\theta}_{\mathfrak{X}}^{-1}(0)$, is a Cohen-Macaulay scheme, not necessarily reduced in general.*

Proof. The dilation action of \mathbb{C}^\times on \mathfrak{t} descends to a contracting \mathbb{C}^\times -action on \mathfrak{t}/W . This makes the map $\tilde{\theta}_{\mathfrak{C}}$ a \mathbb{C}^\times -equivariant morphism. Hence, for any $t \in \mathfrak{t}/W$, one obtains

$$\dim(\tilde{\theta}_{\mathfrak{C}}^{-1}(t)) = \dim(\theta_{\mathfrak{C}}^{-1}(t)) \leq \dim(\theta_{\mathfrak{C}}^{-1}(0)) = \dim \mathfrak{C}^{\text{nil}} = \dim \mathfrak{g} = \dim \mathfrak{C} - \dim \mathfrak{t}/W,$$

Here, the inequality holds thanks to the semi-continuity of fiber dimension, the second equality is a consequence of the (set theoretic) equation $(\theta_{\mathfrak{C}})^{-1}(0) = \mathfrak{C}^{\text{nil}}$, and the third equality is a consequence of (9.1.1). The scheme $\mathfrak{C}^{\text{norm}}$ being Cohen-Macaulay, we conclude that the map $\tilde{\theta}_{\mathfrak{C}}$ is flat and each scheme theoretic fiber of that map is Cohen-Macaulay, cf. [Ma, §16A, Theorem 30].

The proof of the corresponding statements involving the variety \mathfrak{X} is similar and is left to the reader. \square

9.2. We take W -invariant global sections of the sheaves on each side of formula (2.4.1). The functor $\Gamma(\mathfrak{g} \times \mathfrak{t}, -)^W$ being exact, one obtains an isomorphism of left $\mathcal{D}(\mathfrak{g})$ -modules

$$\Gamma(\mathfrak{g} \times \mathfrak{t}, \mathcal{M})^W = \mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g}) \cdot \text{ad } \mathfrak{g}. \quad (9.2.1)$$

Proof of Corollary 1.3.6. The Hodge filtration on \mathcal{M} is W -stable and it induces, by restriction to W -invariants, a filtration F_\bullet on $\Gamma(\mathfrak{g} \times \mathfrak{t}, \mathcal{M})^W$. The functor $\Gamma(\mathfrak{g} \times \mathfrak{t}, -)^W$ clearly commutes with taking an associated graded module. Therefore, we deduce

$$\begin{aligned} \text{gr}^F[\mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g}) \cdot \text{ad } \mathfrak{g}] &= \text{gr}^F[\Gamma(\mathfrak{g} \times \mathfrak{t}, \mathcal{M})^W] = [\text{gr}^{\text{Hodge}} \Gamma(\mathfrak{g} \times \mathfrak{t}, \mathcal{M})]^W \\ &= \Gamma(\mathfrak{g} \times \mathfrak{t}, \text{gr}^{\text{Hodge}} \mathcal{M})^W = \mathbb{C}[\mathfrak{X}_{\text{norm}}]^W = \mathbb{C}[\mathfrak{C}_{\text{norm}}], \end{aligned}$$

where the fourth equality holds by Theorem 1.3.3 and the fifth equality is a consequence of the isomorphism $\mathbb{C}[\mathfrak{C}_{\text{norm}}] = \mathbb{C}[\mathfrak{X}_{\text{norm}}]^W$, see Corollary 1.5.1(i). \square

We recall that the vector space $\mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g}) \cdot \text{ad } \mathfrak{g}$ has a natural right action of the algebra A , cf. §2.4. The isomorphism in (9.2.1) intertwines the natural left $\mathcal{D}(\mathfrak{t})^W$ -action on $\Gamma(\mathfrak{g} \times \mathfrak{t}, \mathcal{M})^W$ and the right A -action on $\mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g}) \cdot \text{ad } \mathfrak{g}$ via the isomorphism $\mathcal{D}(\mathfrak{t})^W \cong A^{\text{op}}$ induced by the map Ξ , see (2.4.1). Therefore, the filtration F_\bullet makes $\mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g}) \cdot \text{ad } \mathfrak{g}$ a filtered $(\mathcal{D}_{\mathfrak{g}}, A)$ -bimodule. In particular, one may view $\mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g}) \cdot \text{ad } \mathfrak{g}$ as a right filtered $(\text{Sym } \mathfrak{g})^G$ -module via the map $(\text{Sym } \mathfrak{g})^G \rightarrow A$ given by the composition of natural maps $(\text{Sym } \mathfrak{g})^G \hookrightarrow \mathcal{D}(\mathfrak{g})^G \rightarrow \mathcal{D}(\mathfrak{g})^G/[\mathcal{D}(\mathfrak{g}) \cdot \text{ad } \mathfrak{g}]^G$.

Proof of Corollary 1.3.7. By Corollary 1.3.6, we have $\text{gr}^F[\mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g}) \cdot \text{ad } \mathfrak{g}] = \mathbb{C}[\mathfrak{C}_{\text{norm}}]$. Therefore, Corollary 1.3.5 implies that $\text{gr}^F[\mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g}) \cdot \text{ad } \mathfrak{g}]$ is a Cohen-Macaulay $\mathbb{C}[\mathfrak{C}]$ -module; moreover, this module is flat over the algebra $(\text{Sym } \mathfrak{g})^G$, by Proposition 9.1.3. Now, a result of Bjork [Bj] insures that $\mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g}) \cdot \text{ad } \mathfrak{g}$ is a Cohen-Macaulay $\mathcal{D}(\mathfrak{g})$ -module which is, moreover, flat over $(\text{Sym } \mathfrak{g})^G$. \square

INDEX OF NOTATION

$T^*X, X_{\text{red}}, X_{\text{norm}}, \psi, \mathcal{O}_X, \mathcal{D}_X, \mathcal{K}_X,$	1.1	p	1.5, 2.1.4
$G, T, \mathfrak{g}, \mathfrak{t}, W, \mathbf{r}, \text{sign}$	1.1	$p_{\text{norm}}, \mathcal{R}, \mathcal{R}^E, E^*, \mathfrak{g}_x, \mathfrak{g}_{x,y}, \mathcal{R}_{\mathbf{x}}, \mathfrak{g}^r, \mathfrak{C}^r$	1.5
$\mathcal{D}, \text{rad}, \text{ad}, \mathcal{M}$	1.2	\mathfrak{g}	1.5, 5.1
$\tilde{\text{gr}}^{\text{Hodge}}$	1.2, 2.5, 4.2.	Small representation	1.6
$\mathfrak{C}, \mathfrak{C}, \mathfrak{T}, \kappa, \text{res}, \langle -, - \rangle$	1.2	L^a, L^g	1.6
v^\top	1.2, 2.4	S_n	1.6, 1.9
\mathfrak{X}	1.2, 2.1.4	$\mathbf{e}, \bullet_{\mathbf{e}}, h_s, \mathbf{h}, \mathfrak{g}^s, R_s^+, W_s, \mathbf{d}_s$	1.7
R^+	1.4, 5.2	$\mathbb{C}^{\leq m}[\mathfrak{t}], 'F_\bullet, ''F_\bullet,$	1.7, 7.1, 7.2

$\mathrm{gr}^F \mathbb{C}[W \cdot \mathbf{h}]$	1.7	\mathfrak{z}	3.3, 4.2
Exceptional principal nilpotent pair	1.8	π	3.3.5, 4.5
$\check{h}_s, \Delta_e, \mathbf{s}_e, \mathbf{s}_e(e), \mathbf{E}, \mathbf{S}$	1.8	$\mathcal{A}, \mathcal{H}^j, \mathrm{eu}, \partial_{\bar{\kappa}^* \mathrm{eu}}$	3.4
$\mathrm{Hilb}^n(\mathbb{C}^2), \widetilde{\mathrm{Hilb}}^n(\mathbb{C}^2), \mathfrak{C}^\circ, \mathcal{P}, A^k$	1.9, 8.1	ϖ, ς	(4.2.1)
\mathcal{V}	1.9, 8.3.1	$\mathcal{G}_X, \mathbf{g}_X, L^{\mathbf{g}_X}$	5.1
$\mathfrak{g}^{rs}, \mathfrak{C}^{rs}, \mathfrak{X}^{rs}, \mathfrak{g} // G, \mathfrak{C} // G, \mathfrak{x}, \mathfrak{x}^{rs}, \mathfrak{r}, N_{\mathfrak{r}}$	2.1	$R, \mathbb{X}, \preceq, L_{\mathfrak{b}}^{\preceq \lambda}, L_{\mathfrak{B}}^{\preceq \lambda}, L_{\mathfrak{B}}^{\lambda}, \mathfrak{g}_{\mathfrak{B}}^{\alpha}, \mathfrak{h}, L^{\mathfrak{h}}$	5.2
$p_{\mathfrak{x}}$	2.1.4	$pr, \tilde{\mathfrak{g}}^r, \gamma, \tilde{\gamma}, \vartheta, L_{\mathfrak{g}}^{\lambda}, \mathfrak{g}_{\mathfrak{g}}^{\alpha}$	5.3
$q, F^{\mathrm{ord}}, \mathrm{gr}^{\mathrm{ord}}, \tilde{\mathrm{gr}}^F, \mathcal{D}_{X \rightarrow Y}, \int_f, \int_f^R$	2.2	$L_{\mathfrak{r}}^{\lambda}, \lambda^L, \lambda^k$	5.4
$A, \Xi, \mathcal{I}, \mathcal{J}, F^{\mathrm{ord}} \mathcal{M}, \mathcal{J}$	2.4	$L_{\mathfrak{r}}^{\lambda}, \mathfrak{g}_{\mathfrak{r}}^{\alpha}, e_{\alpha}^k$	5.7
$\mathfrak{r}^r, \delta_{\mathfrak{t}}, dx, dh, \mathbf{j}, \mathbf{j}_* \mathcal{O}_{\mathfrak{r}}$	2.4.2	$h_x, n_x, \mathfrak{G}^{\mathfrak{b}}$	6.1
$F\mathcal{D}_X\text{-mod}, D_{\mathrm{coh}}^b(F\mathcal{D}_X), \tilde{\mathrm{gr}}(M, F)$	2.3	$\mathfrak{C}(\mathfrak{l}), \mathfrak{N}(\mathfrak{l}), S, \mathfrak{t}_{\mathfrak{l}}, \mathring{\mathfrak{t}}_{\mathfrak{l}}, \mathring{\mathfrak{X}}_{\mathfrak{l}}, \mathfrak{X}_{\mathfrak{l}}(\mathfrak{g})$	6.2
Hodge module, Hodge filtration	2.3	$\mathfrak{C}_{\mathfrak{l}}(\mathfrak{g})$	6.3
$\mathrm{HM}(X), \mathbb{D}, F\mathbb{D}, \mathrm{Coh} Z, D_{\mathrm{coh}}^b(Z), \widetilde{\mathrm{dgr}},$	2.3	$RE, 'R, ''R, 'gr, ''gr, \mathfrak{N}, F^{\min}, F^{\max}, \mathrm{can}$	7.1
F^{Hodge}	2.5, 4.2	$\mathbb{C}^{i,j}[\mathfrak{T}], I_{\mathfrak{h}}, \mathfrak{Y}, \wp, I_{\mathfrak{k}}$	7.2
$\mathcal{H}_Z^k, \mathfrak{C}_i, \mathfrak{C}^{rr}, \mathfrak{X}_i, \mathfrak{X}^{rr},$	2.6	$\varkappa, \tilde{\varkappa}, \Upsilon$	7.3
$\mathcal{B}, \mathfrak{b}, \tilde{\mathfrak{g}}, \tilde{\mathfrak{G}}, \nu, \boldsymbol{\nu}, \mu, \boldsymbol{\mu}, \mathcal{T}_X,$	3.1	$S^m, \hat{S}, H^{\perp}, \delta_s, V_s, \check{V}_s, \nabla, \nabla_{i,j}$	7.5
π, ω	3.1.1	$f^{\#}, \mathrm{res}_e, u, \mathrm{pr}_{\mathcal{R}}^{\mathrm{sign}}, \mathrm{pr}_{W \cdot \mathbf{h}}^{\mathrm{sign}}, \mathbf{E}_{\mathbf{x}}$	7.6
q	3.1.1, 3.4	$G_{x,y}, \mathbb{C}\langle x, y \rangle, \mathcal{S}, \mathrm{Hilb}, \rho, \delta$	8.1
ϵ, Λ	3.2, 4.3	$\mathfrak{W}, \mathfrak{X}^\circ, \eta, \delta, \tilde{\rho}, \tilde{p}, h, \varrho$	8.2
$\mathrm{pr}_{\Lambda \rightarrow T^* \mathcal{B}}, \mathrm{pr}_{\Lambda \rightarrow \mathfrak{G} \times \mathfrak{T}}, \mathcal{N}, \tilde{\mathcal{N}}, \Phi, \Psi, \tilde{\kappa}$	3.2	$\mathrm{Map}_G, \mathcal{L}_Y(E), \alpha, \alpha_i, \Phi_i, \beta_S, \beta_V$	8.3.1
$\tilde{\mathfrak{X}}, \tilde{\mathfrak{X}}^{rr}$	3.3	$Y, \mathbf{D},$	8.4
		$\mathfrak{C}^{\mathrm{nil}}, \theta_{\mathfrak{C}}, \theta_{\mathfrak{X}}, \tilde{\theta}_{\mathfrak{C}}, \tilde{\theta}_{\mathfrak{X}}$	9.1

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